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MATHEMATICS

Mechanics
Unit M1

W E Williams & S Y Barham

AS/A LEVEL

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WJEC AS/A Level Mathematics
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FOREWORD

Professor Williams was the Chief Examiner for the Mathematics A2 paper and the Applied Mathematics modular papers for the Welsh Joint Education Committee and was also the Chief Examiner for a number of other Boards.

Dr Barham is the current Chief Examiner for the Mathematics A2 paper and the Applied Mathematics modular papers, also the new modular papers for the Welsh Joint Education Committee.

This book covers the new M1 syllabus of the Welsh Joint Education Committee and only assumes the Pure Mathematical knowledge covered in the P1 syllabus.

An attempt has been made to present the material in such a way that students can make considerable headway before needing some of the slightly more sophisticated Pure Mathematical concepts.

In Chapters 2 and 3 only the elementary properties, associated with a right angled triangle, of the trigonometric functions have been assumed.

The material in Chapter 4, other than 4.4, should be accessible without any knowledge of calculus. Similarly all the material in Chapters 6 and 7 is not dependent on calculus.

We are indebted to a number of teachers who made various suggestions. In particular, we extend our thanks to Kevin McGuire, John Langley and Elwyn Davies who checked and sent us corrected answers to exercises.

Every effort has been made to eliminate errors present in previous versions of the book. However, any that remain are the responsibility of the authors.

CONTENTS

Chapter 1

Mechanics and Modelling

After working through this chapter you should

- have some idea of the principles of modelling in Mechanics,
- be aware of some of the limitations of the particle model.

1.1 Basic principles of modelling

Mechanics is basically the study of what causes bodies to move and how they move. Statics deals with bodies at rest and Dynamics with bodies in motion. A knowledge of Mechanics is essential to design many of the things occurring in everyday life e.g. cars, aeroplanes, bridges, roads, houses. It is important, particularly when designing something new and complicated like a bridge or a car, to have some idea of whether the design will work. It is a bit late when the bridge is built to find a flaw!

The easiest way of finding whether the design is satisfactory is to try and formulate the real problem in mathematical terms and then use mathematical methods to predict what happens. This is what is known as constructing a mathematical model of a real situation and a mathematical model is effectively a simplified representation of a real world problem in mathematical terms. Real problems are often very complicated and in order to be able to do some useful mathematics it is usually necessary to make many simplifying assumptions. The idea is only to make simplifications which still retain the basic features of the problem. Once the assumptions have been made, mathematical calculations can be carried out and the predictions compared with any available experimental data.
If there is good agreement between theoretical predictions and experiment then the model has provided a good realisation of the real problem and a satisfactory solution has been obtained. This is the outer loop in the following diagram.

Identifying the problem → Formulating a model → Obtaining a mathematical solution to the problem

↑

Refining the model

↑ ↓

No

↑

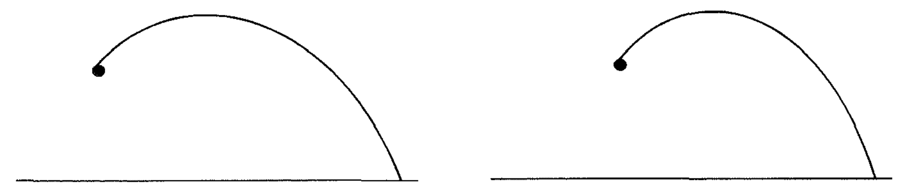
Communicate results ← Yes ← Do results agree with observation ← Interpreting the solution

If the model does not agree with the observation then the model has to be changed (refined) and the cycle repeated, possibly several times. This is the inner loop in the above diagram.

The advantage of a mathematical model is that, once it has been shown to be valid for a range of parameters, it is possible to predict what happens, without further experiment, when parameters are changed. For example, a mathematical model could be constructed for the motion of the bumper of a car when the car was involved in a crash. This model would involve a parameter describing the behaviour of the springing between the bumper and the car body. The model could be verified by crash tests for one or more types of springing, and then the behaviour of the bumper for various types of springing determined, and the design incorporated in new cars without further tests to destruction.

In situations such as designing a new bridge or aeroplane, scale models would be built to compare the calculation with experiment. In many standard engineering applications however, the modelling assumptions are generally well understood and have stood the test of time and in these cases the mathematical model can be used to produce particular design characteristics without the need for further tests.

In your Mechanics course you will have to model fairly simple situations by making standard assumptions and you will have to be aware of the implications of these assumptions. For example you may be given the problem of a shot-putter throwing the shot at a certain speed and asked to find where the shot hits the ground. You will learn that if the shot is modelled as a particle and the acceleration due to gravity is assumed to be constant, then a fairly simple solution can be found. The path predicted by the model will be that shown by the left hand diagram, the actual path will be something like that in the right hand diagram.



It may not be obvious which of the modelling assumptions should be changed in order to obtain agreement with observation. You are generally aware that, as you walk and run, you experience some air resistance and therefore it might be reasonable to refine the model to take this into account. This is not particularly easy and different types of air resistances have to be considered. It is possible to eventually arrive at a more accurate model but the calculations are complicated. You should however be aware that, when air resistance is taken into account, the predicted horizontal distance travelled by the shot is decreased. Calculations on the more complicated model show that under typical conditions the error in neglecting resistance is about 3%. In practice, neither the speed of the throw, nor the angle of projection, would be known to this level of accuracy and therefore it is pointless to try and construct an elaborate model in these circumstances. It is important, when modelling, not to try and set up an elaborate model when the data to be used in the model is not particularly accurate.

There are circumstances where extreme accuracy is necessary and where the data is known accurately. An example of this is the free flight path of a space shuttle returning to earth. In this case it is necessary to assume, to get the required level of accuracy, the exact form of the acceleration of the earth, and also take into account corrections due to the rotation and the curvature of the earth.

In the Glossary in Chapter 2 most of the standard assumptions generally made are described and a more detailed account of the assumptions is given in 2.4.

There is one particular modelling assumption which is made throughout most of the book and this is modelling a body as a particle and the assumptions and limitations of this model will now be considered.

1.2 Implications of particle model

In practically all the problems that you will come across in your course, and certainly in all problems involving motion, all bodies will be modelled as particles. A particle is effectively something with no size but with mass or weight. You will be given the precise definitions of these in subsequent chapters but you will already have some rough idea of weight. Effectively a particle is represented by a point. There are two basic assumptions implicit in using the particle model. The first assumption is an essentially geometric one that the dimensions of the body are small compared with the other dimensions involved in the problem. Therefore, for example, in tracking an aircraft it is reasonable to represent it by a point, since knowing the position of a point of an aircraft would give you a very good idea of the position of the whole aircraft. This is roughly what happens on radar screens tracking aircraft and ships. In the context of space, the aircraft is very much smaller than the distance from a tracking station. Similarly in analysing the motion of the earth around the sun, it is sufficient, in view of the distances involved, to represent the earth as a particle.

It is very important to realise, when modelling, that the model you need to choose on a particular occasion depends on the occasion and on what you are trying to find. For example, a particle model of a car is sufficient to give information about its position and speed but it would be completely inadequate for road design. In the latter case it is necessary to calculate the overtaking sight distance (the distance open to view of a car travelling at the design speed of the road and wanting to overtake slower traffic without causing an obstruction) and so the model would have to take into account the lengths of vehicles.

The other assumption of the particle model of a body is that effectively the motion of one point of the body is completely representative of the motion of the whole body. This is not always true as you can see, for example, by throwing up a paper plate. All the points eventually move downwards but there will be a considerable amount of wobbling and certainly not all points of the plate move in the same way. It is possible to prove that the general motion of a body involves both a direct motion (translation) and a rotation. You can see this by throwing up a ball, since in most circumstances there will also be a rotation or spin.

The particle model completely ignores the fact that not all points move in the same way and so is insufficient to model some situations. To a large extent, the motion parallel to the road of all points of a car is the same and therefore the particle model is adequate. There will however be some slight motion perpendicular to the road, due to the effect of the suspension, and a model involving a box attached to four springs would be necessary to analyse this motion.

A simple example of the inadequacy of the particle model is when you have a tall piece of furniture on a rough floor and you push it near the top. The particle model predicts that the furniture slides as one piece but the reality is that it sometimes topples!

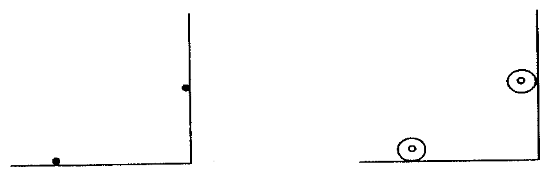
Another example is the motion of a snooker ball. The particle model would always predict that the ball would go in a straight line but in actual fact it can be made to go in a curve. A model taking spin into account predicts the curved motion. You may have seen changing order golf balls struck into a hole and then travel round the inside of the hole before coming out, this would not be predicted by the particle model.
The saving grace of the particle model for problems involving motion is that there does exist for any body a point, whose motion is exactly that of a particle of mass equal to that of the body and acted on by all the external forces acting on the body. For most balls this point will be at its geometric centre. Therefore the particle model will give a very good estimate of where a ball lands but it will give very little information about what happens afterwards because, as you have seen in ball games, the spin on the ball has an enormous effect on the path after impact.

1.3 Refinements of modelling

In order to refine a model you have to be very clear what your initial modelling assumptions are and then see which of them can, or need, to be changed. The most frequent modelling assumption is to ignore friction or air resistance and, as you will find later, it is often possible to refine, fairly easily, the model to take these into account. Even when you are not required to carry out detailed calculations you should be aware of the qualitative implications of the refinement. For example, if a ball is thrown vertically upwards, the maximum height it reaches will be less than that predicted by a model ignoring air resistance. The latter model therefore overestimates the maximum height reached. This point will be explained in more detail in Chapter 5.

The modelling assumptions used in constructing the table of stopping distances in the Highway Code assume that the road is flat. The model has to be refined to take into account whether there is an uphill or downhill slope. The particle model of a car shows that the maximum slope on which a car can be parked without slipping is independent of whether the front wheels are pointing down or up the hill. A more accurate model which takes the car as a box on wheels shows that this is not always the case.

It is sometimes relatively easy to refine the particle model so as to take some account of the size of a body. For example, you will learn later in the course how to find the time taken for a ball (modelled as a particle), struck from a floor, to a hit a vertical wall at right angles, as shown in the left hand diagram below.



If the ball is of radius a you can take the size into account by representing the ball by a particle starting at a point at a distance a above the floor, as in the right hand diagram, and finding the time until it is at a horizontal distance a from the wall.

It is however difficult to take account of the spin of a body in such a relatively simple way.

Chapter 2

Forces acting at a point

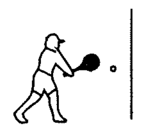
After working through this chapter you should

- have a clear idea of what a force is,
- understand what is meant by the resultant of several coplanar forces acting at a point and be able to find the resultant and its components in two perpendicular directions,
- appreciate some of the ideas used in modelling the effects of forces on small bodies,
- be able to solve problems relating to the equilibrium of a small body,
- be able to apply the laws of friction in simple cases.

2.1 Forces

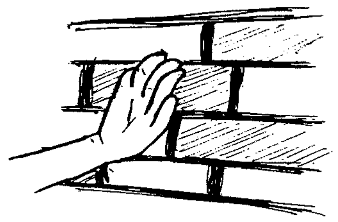
Most people have an intuitive idea of a force as a "push" or a "pull". Taking this idea further, pushing or pulling a book on a desk, for example, will make the book move. Therefore, in this case, a force has caused a change in the motion of the book.

There are many other instances of what would be instinctively regarded as a force producing a change in motion, for example a tennis player hitting a ball or a driver braking a car.

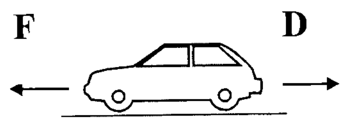


The idea of a force as something which changes the motion of a body actually defines a force. On this definition most, but not all, forces conform to the idea of a "push" or a "pull". There are essentially two classes of forces:- contact forces where the change in motion is produced by direct contact, and non-contact forces where there is no direct physical link to the body. The most obvious example of the latter is the force due to gravity: this does however conform to the simple idea of a "pull" since if you jumped from a wall, you would certainly experience a pull!

Though a force is defined as "that which changes the motion of a body" it is not always true that applying a force will change the motion. For example pushing a brick wall will not bring it down.



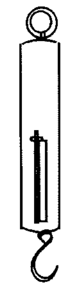
This is because there are other forces binding the wall together and you cannot push sufficiently hard to overcome them. If a heavy lorry hits a wall then the chances are high that the wall would move. When you push a wall you will feel a resistance, i.e. the wall pushes back. This is an example of Newton's third law which states that "to every action there is an equal and opposite reaction". The tennis player also feels this reaction when she hits the ball.
Another example where you may think that there are forces acting but there is no change in motion is a car moving at a steady speed.



There is a driving force F exerted by the wheels but there is also a drag force D due to air resistance and when the speed is not changing, the forces balance, i.e. the net force is zero.

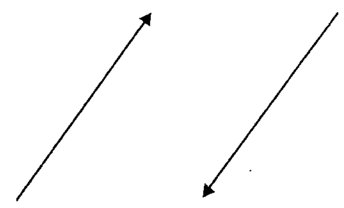
The exact relation between force and motion is provided by Newton's second law of motion (5.1), which states that the force acting on a moving particle is proportional to the particle's acceleration.

Measuring force
One of the simplest ways of measuring some contact forces is by using a spring balance. This is a device where the top end of a spring is fixed, a hook is attached to the bottom end and the spring is allowed to hang freely. The diagram shows an old fashioned type of spring balance that you may still sometimes see and you probably have seen smaller balances in your science courses or Mechanics kits.



If you pull down on the hook then the spring extends and a small needle attached to it moves down: if you pull harder then the needle drops further. The spring balance is a good method of measuring force since, for forces of reasonable size, the extension of the spring is directly proportional to the force. (This is Hooke's law which you will meet in 2.2). The balance can be calibrated by taking a particular force as the unit of force and the extensions corresponding to different forces are then marked out on the scale. The spring balance is of limited use in measuring forces in practical situations but it is important, for theoretical reasons, to know that a unit of force can be defined independently of motion.
The most commonly used unit of force is the newton abbreviated to N, so that a force of six newtons would be written as 6 N. The formal definition of the newton is given in (5.1). A thousand newtons is denoted by 1 kN.

Representing forces
If a small body is attached to a thin rod and the other end of the rod is pulled, then the body will move in the direction of the rod. The more effort exerted on the rod, the more rapid the motion. If the rod is pulled in a different direction then the body will move in a different direction. If, in either case, the rod was pushed, not pulled, the motion would be in the opposite direction.
The rod, since it produces a change in motion, exerts a force on the body and this force has both magnitude (more effort produces faster motion) and direction (the different directions of the rod). Therefore a force can be represented by a directed line, with the length of the line representing the magnitude of the force and its direction, usually shown in diagrams by an arrow head, being that of the force. Since pushing and pulling produces different motions, it is important to show the direction along the line in which the force is acting. The two forces shown, though acting along parallel lines and equal in magnitude, are different from each other since their directions are opposite.



Quantities having both magnitude and direction are called vectors and you will learn more about them in book M2.

In order to distinguish, in print, between a force and its magnitude bold type will be used to refer to the force and italic type for the magnitude. Therefore **F** refers to a force and *F* to its magnitude. You could do this in writing by underlining when you refer to the force and not when you are referring to its magnitude. If a force **G** is equal in magnitude to another one **F** and acts along the same, or parallel, line but in the opposite direction, then **G** could be denoted by -**F**. Therefore if the left hand force in the above diagram is **F** then the right hand force is -**F**.

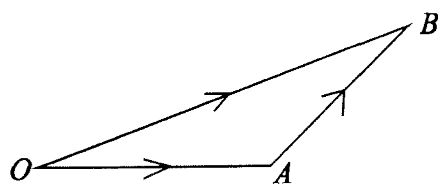
Combining forces

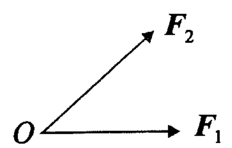
If two rods were attached to a body and both rods were pulled then, as long as the pulls were not equal and opposite, the body would still move. The direction of the motion would not usually be parallel to either rod, but somewhere in between them. There is therefore a force acting on the body. This is called the resultant of the two forces and it is a single force that has the same effect as the two forces. Obviously more than two rods could be attached and :

The resultant of any number of forces acting at a point is the single force which has the same effect (i.e produces the same motion) on a small body placed at that point.

Resultant of two forces acting at a point

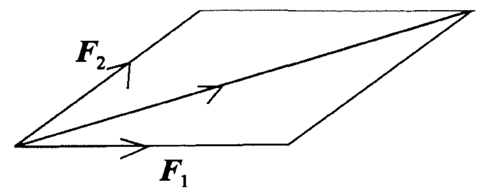
The rule for finding the resultant of two forces is a slightly unusual one and is as follows.





If there are two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ acting at the point O as shown and if the line OA represents $\boldsymbol{F}_1$ and the line AB represents $\boldsymbol{F}_2$, the line OB represents the resultant of the two forces. This is, for obvious reasons, often called the "triangle rule" for combining (or adding) forces.

An alternative way of expressing this rule is to say that the resultant is represented, as shown above, by the diagonal of the parallelogram formed by the lines representing the two forces (this is the "parallelogram rule").



This rule can be shown to be a consequence of Newton's law of motion but it can also be verified experimentally. You may have seen a verification carried out in one of your science courses. A possible experimental set-up is described in Miscellaneous Exercises 2, q 23.

For two parallel forces the rule simplifies:

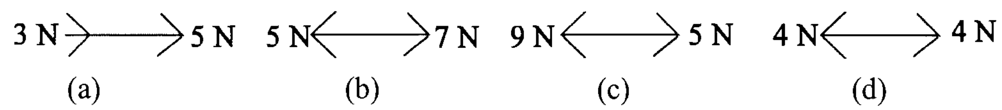
(a) for two forces acting in the same sense, the resultant acts in the same sense and its magnitude is the sum of the magnitudes;

(b) for two forces acting in the opposite sense, the resultant acts in the sense of the force with the greater magnitude and its magnitude is the positive difference of the magnitudes;

(c) for two forces of equal magnitude, but of opposite direction, the resultant force is zero. (You can check this rule by trying particular cases).

<u>**Example 2.1**</u>

Find the resultant of the following systems of forces:



(a) Both forces are acting in the same direction so the resultant acts to the right and is of magnitude 8 N.

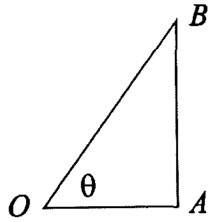
(b) The forces are acting in the opposite direction, the force acting to the right has the greater magnitude and therefore the resultant acts to the right and is of magnitude 2 N.

(c) The forces are acting in the opposite direction, the force acting to the left has the greater magnitude and therefore the resultant acts to the left and is of magnitude 4 N.

(d) The forces are acting in opposite directions, both have the same magnitude and therefore the resultant is zero.

Example 2. 2

There are two perpendicular forces of magnitudes 3 N and 4 N acting at a point O. Find their resultant.



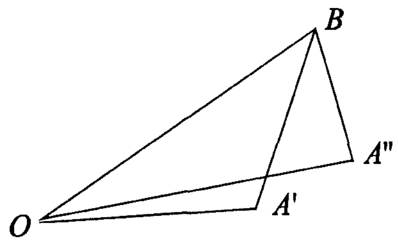
In the above diagram OA represents the force of magnitude 3 N and AB represents the force of magnitude 4 N. Their resultant will be represented by OB. By Pythagoras' theorem its magnitude will be $\sqrt{3^2 + 4^2}$ N = 5 N. The angle θ is given by

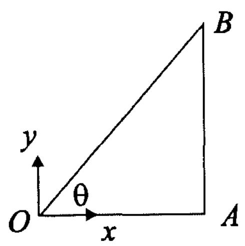
$$\tan\theta = \frac{4}{3}.$$

Using the $\tan^{-1}$ function on your calculator gives θ to be approximately 53.1°.

Example 2.2 shows that a force represented by OB (5 N acting at an angle of approximately 53.1° to a line going across the page from left to right), is the resultant of two perpendicular forces of magnitudes 3 N and 4 N respectively. These two forces are referred to as perpendicular components of the force along OB.

Components of a force





In the left hand diagram above, OB represents a given force $\boldsymbol{F}$. An infinite number of triangles can be drawn from points A', A'', A''' etc onto OB as base so that $\boldsymbol{F}$ can be regarded as the resultant of forces represented by OA' and $A'B$, or OA'' and $A''B$, etc. Therefore there are an infinite number of ways in which $\boldsymbol{F}$ can be expressed in terms of two separate forces or components. The process of representing a force in terms of two

components is referred to as resolving the force into its components (or into its resolved parts).

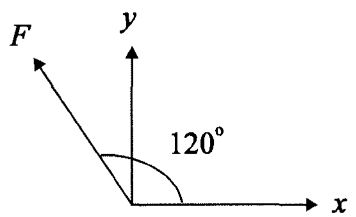
The most usual way of resolving a force is into components along two perpendicular lines which gives a right angled triangle such as OAB in the right hand diagram above. For ease of reference, the x and y axes of a coordinate system are chosen to be in the directions of OA and AB respectively. The component in the x-direction (called the x-component) of the force $\boldsymbol{F}$ represented by OB is denoted by X and the component of the force in the y-direction (i.e. the y-component) by Y. The triangle rule for resultants states if the length of OA is proportional to X and the length of AB is proportional to Y then OB is in the direction of $\boldsymbol{F}$ and its length is proportional to F.

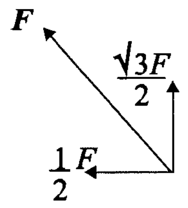
The angle between the x-direction and $\boldsymbol{F}$ is denoted by θ. The triangle OAB is right angled and you know, from the definitions of sine and cosine, that

$\dfrac{OA}{OB} = \cos\theta$, and therefore $\dfrac{X}{F} = \cos\theta$, i.e. $X = F\cos\theta$.

Similarly $\dfrac{AB}{OB} = \sin\theta$ so that $\dfrac{Y}{F} = \sin\theta$, i.e. $Y = F\sin\theta$.

The component of a force $\boldsymbol{F}$ in a given direction can now be defined as $F\cos\alpha$, where α is the angle between $\boldsymbol{F}$ and the given direction. This is consistent with the above results since the angle between $\boldsymbol{F}$ and the positive x-direction is θ and the angle between $\boldsymbol{F}$ and the positive y-direction is $\frac{\pi}{2} - \theta$, giving the y-component as $F\cos(\frac{\pi}{2} - \theta) = F\sin\theta$, since $\cos(\frac{\pi}{2} - \theta) = \sin\theta$. This definition of component also shows that, when the angle between the force and the reference direction is obtuse, the component can be negative.





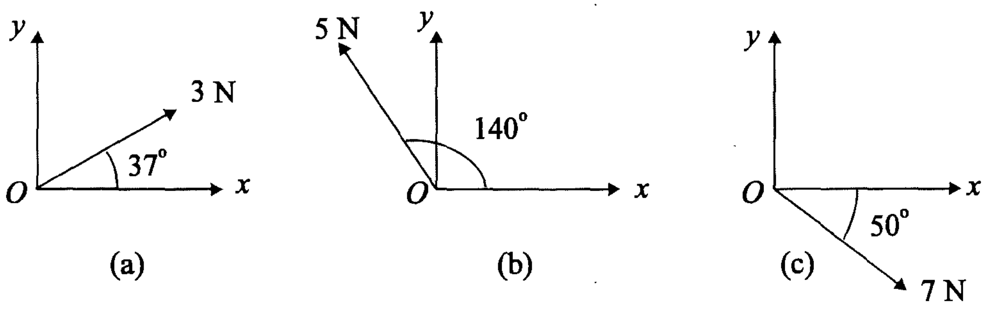
This occurs in the diagram above where $\theta = 120°$ so that $X = F\cos(120°) = -\frac{1}{2}F$ and

$Y = F\cos(30°) = \dfrac{F\sqrt{3}}{2}$. All this means is that the force is the resultant of $\frac{1}{2}F$ to the left and $\dfrac{F\sqrt{3}}{2}$ up the page.

You need to be very careful about the signs of the components. One way of doing this is by being very careful to pick the correct angle and take its cosine correctly. Another way is to resolve the force into two positive perpendicular components along the lines in which you are required to find components. This is shown in the diagram above. It is worth remembering that if θ is the acute angle with one of the lines, then the positive components along the lines are $F\cos\theta$ and $F\sin\theta$. If the directions of these components are opposite to the actual direction in which you have to find the component then the component in the required direction is found by changing the sign. In the above diagram the component to the left is $\frac{1}{2}F$ and so the component to the right is $-\frac{1}{2}F$. When you start to find components it is a good idea to show both the force and its positive components as described above.

<u>Example 2.3</u>

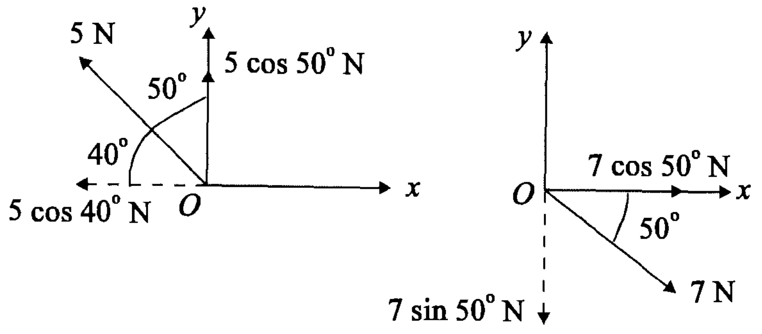
Find the x- and y- components of the following forces.



(a) The angles between the force and the x and y axes are 37° and 53° respectively, so the components are 3 cos (37°) N = 2.40 N and 3 cos (53°) N = 1.81 N.

(b) The angles between the force and the positive x and y directions are 140° and 50°, respectively, so the components are 5 cos (140°) N = −3.83 N and 5 cos (50°) N = 3.21 N.

Alternatively the positive components along the x- and y- axes are shown in the following left hand diagram.



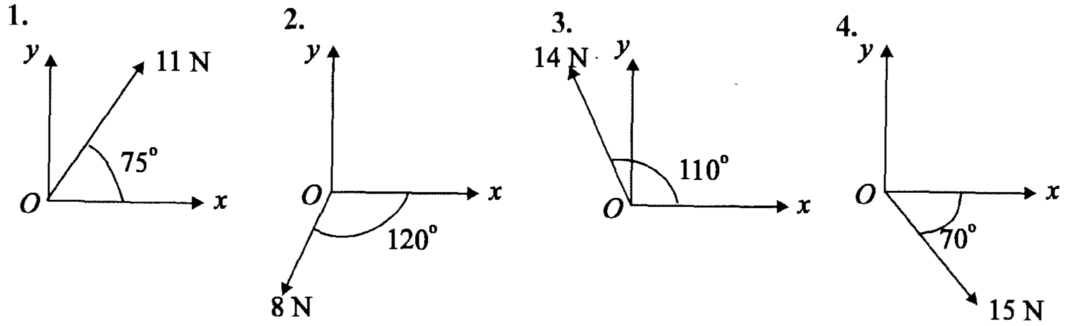
The components to the left and upwards are 5 cos (40°) = 3.83 N and 5 cos (50°) N {or 5 sin 40° N} = 3.21 N. Therefore the x- and y- components are = −3.83 N and 3.21 N.

(c) The angles between the force and the positive x and y directions are 50° and 140°, respectively, so the components are 7 cos (50°) N = 4.50 N and 7 cos (140°) N = −5.36 N. Alternatively the positive components along the x- and y- axes are shown in the right hand diagram above.

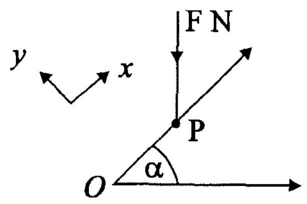
The components to the right and down are 7 cos (50°) N = 4.50 N and 7 sin (50°) N = 5.36 N. Therefore the x- and y- components are 4.50 N and −5.36 N.

<u>Exercises 2. 1</u>

In all numerical questions, answers should be given to three significant figures. Find the x- and y- components of the forces shown in questions 1 to 4.



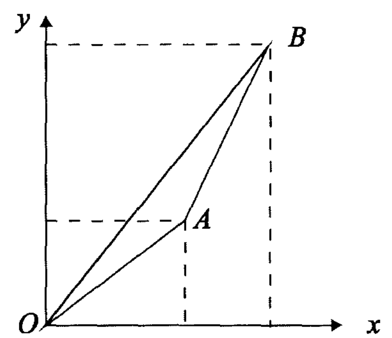
5.



The diagram represents a section of a plane inclined at an angle α to the horizontal and a force of magnitude F N acts vertically downwards at a point P of the plane. Find the x- and y- components of this force, referred to the axes shown which are parallel and perpendicular to the plane, when
(a) $F = 10$, $\alpha = 30°$, (b) $F = 6$, $\alpha = 50°$.
6 Carry out the same calculations as in 5 when the vertical force is replaced by a horizontal one acting to the right and of magnitude Q N when
(a) $Q = 8$, $\alpha = 60°$, (b) $Q = 4$, $\alpha = 70°$.

Components of the resultant of forces acting at a point

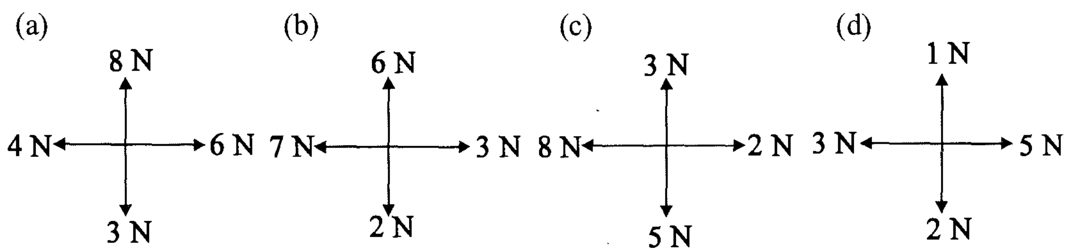
For several forces acting at a point, the component of their resultant in a particular direction is the sum of the components of the separate forces in that direction.
No attempt will be made to prove this result in general but you can see, from the following diagram, that it is certainly true for two forces in the simple case shown.



In Examples 2.4 to 2.6, the x-direction is taken to be to the right across the page and the y-direction to be up the page.

Example 2.4

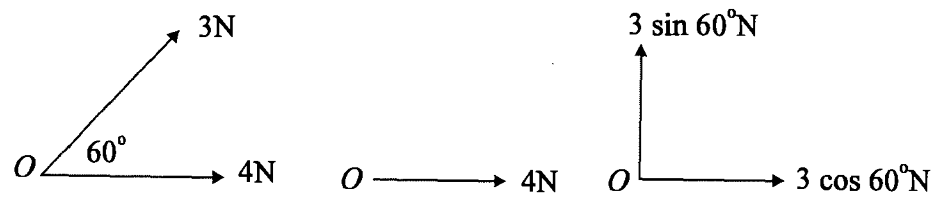
Find the x- and y-components of the resultant of the following forces acting at a point.



(a) The total x-component is 6 N–4 N = 2 N, whilst the y-component is 8 N–3 N = 5 N.
(b) The total x-component is 3 N–7 N = –4 N, whilst the y-component is 6 N–2 N = 4 N.
(c) The total x-component is 2 N–8 N = –6 N, whilst the y-component is 3 N–5 N = –2 N.
(d) The total x-component is 5 N–3 N = 2 N, whilst the y-component is 1 N–2 N = –1 N.

Example 2.5

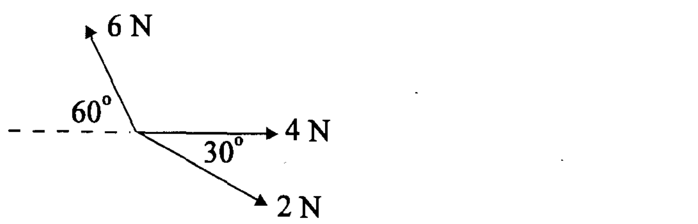
Find the x- and y-components of the resultant of the forces of magnitude 3 N and 4 N, acting at the point O, as shown in the left hand diagram below.



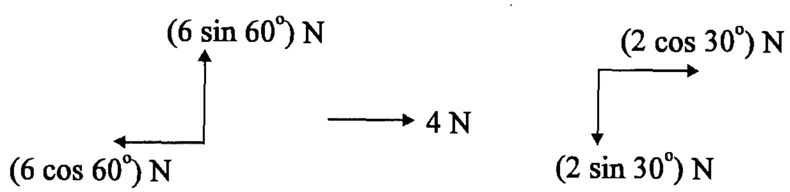
The right hand diagrams show each force resolved into its positive components across and along the page. The x- and y- components of the resultant therefore are $4 + 3 \cos 60°$ N = 5.5 N and $3 \sin 60°$ N = 2.60 N.

Example 2.6

Fnd the x- and y- components of the resultant of the following forces acting at a point O.



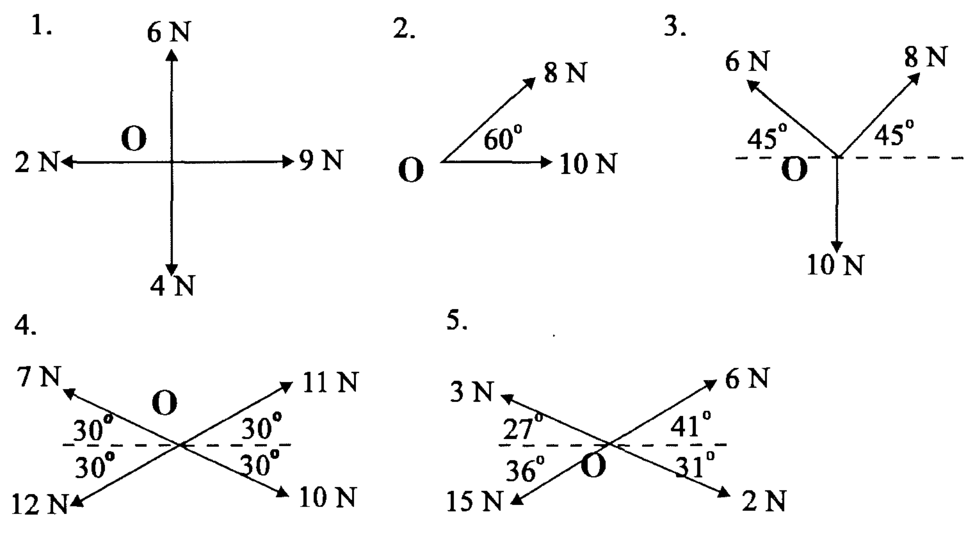
Each force can be resolved into components as shown below .



The force of magnitude 6 N has a negative component in the positive x- direction of $-6\cos 60^\circ$ N, whilst the force of magnitude 2 N has a negative component in the y-direction of $-2\sin 30^\circ$ N.
The x- and y- components of the resultant are therefore $2\cos 30^\circ + 4 - 6\cos 60^\circ$ N = 2.73 N and $6\sin 60^\circ - 2\sin 30^\circ$ N = 4.20 N.

Exercises 2.2

In each of the following problems find the x-and y- components of the resultant of the forces shown acting at a point O.



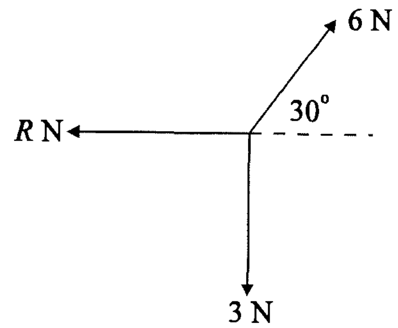
Equilibrium problems

When a number of forces acting at a point have zero resultant, the forces are said to be in equilibrium. Since a point (particle) is a model for a small body you can say that, if the resultant force is zero, the point (particle, body) is in equilibrium. A body in equilibrium is at rest.
Forces acting on a particle will be in equilibrium if the sum of the components, in two non parallel directions, of all the forces acting is zero.
(This follows from the triangle rule since if a triangle has two sides zero, its third side must also be zero.)
The condition that components in two directions are zero for equilibrium is the one that is generally easiest to apply in problems and it is used in Examples 2.7 to 2.9. For problems involving only three forces, there is an alternative geometric method which is sketched after Example 2.9.

In order to have some practice with the general technique it is useful to look first at a few problems which are basically numerical. Practical problems requiring modelling and the use of given physical conditions are given in 2.2.

Example 2.7

Find R such that the forces shown below are in equilibrium.



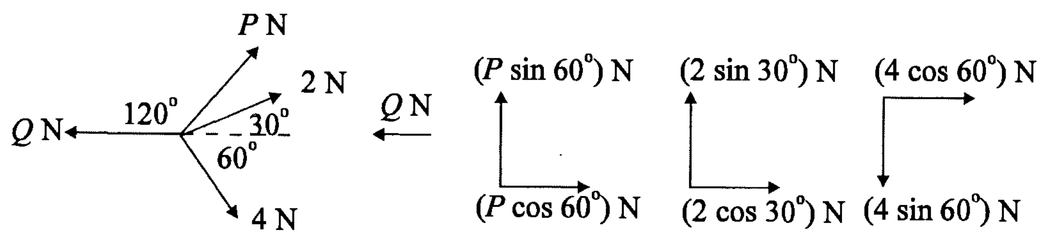
The force of magnitude 6 N has components $6\cos 30^\circ$ N and $6\sin 30^\circ$ N to the right and up the page respectively.
The unknown force has a component of magnitude R N to the left and the third force has a component of magnitude 3 N down the page. Paying attention to the senses of the various forces, the components of the resultant to the right and up the page are $6\cos 30^\circ - R$ N and $6\sin 30^\circ - 3$ N = 0 N. The component up the page is already zero and, therefore, for equilibrium, the other one must also be zero so

$$R = 6\cos 30^\circ \text{ N} = 3\sqrt{3} \text{ N}.$$

An alternative solution to this example is given after Example 2.9.

Example 2.8

Find P and Q such that the system of forces shown in the left hand diagram below is in equilibrium.



The separate forces have components as shown in the right hand diagram. On taking account of the directions of the components, the x- and y- components of the resultant are ($P\cos 60^\circ + 2\cos 30^\circ + 4\cos 60^\circ - Q$) N and ($P\sin 60^\circ + 2\sin 30^\circ - 4\sin 60^\circ$) N respectively.

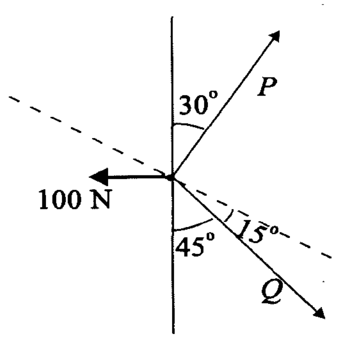
Both components have to be zero. Equating the second one to zero gives $P\sin 60^\circ = 4\sin 60^\circ - 2\sin 30^\circ$ so that $P = 2.85$. Equating the first component to zero gives $Q = P\cos 60^\circ + 2\cos 30^\circ + 4\cos 60^\circ$ and, finally substituting for P, $Q = 5.16$.

As you get more familiar with using components you will not need to use separate diagrams to work out the components of the various forces and should be able to carry out the calculations in your head. It is also a matter of preference whether you work out the total component in a given direction and equate it to zero or, for example, equate the components to the left (or up) to those to the right (or down). The "shorthand" for saying what you are doing is resolving parallel and perpendicular to a given direction (you should always say clearly the direction in which you are resolving).

The choice of the directions in which to resolve is yours and you can sometimes make life easier by resolving perpendicular to an unknown force since this force will not have a component perpendicular to itself since $\cos 90^\circ = 0$.

Example 2.9

Find the values of P and Q so that the system shown below is in equilibrium.



Resolving perpendicular to the force of magnitude 100 N gives

$$P\cos 30^\circ = Q\cos 45^\circ,$$

resolving parallel to the above force gives

$$P\cos 60^\circ + Q\cos 45^\circ = 100 \text{ N}.$$

These equations have now to be solved for P and Q and the final answers are

$$P = 73.2\text{ N},\ Q = 89.7\text{ N}.$$

This problem can actually be solved without getting two simultaneous equations, by resolving along the perpendiculars to the unknown forces. The dashed line is perpendicular to the force of magnitude P and the forces of magnitudes Q and 100 N make angles of 15° and 30° with this line. Resolving parallel to the dashed line gives

$$Q\cos 15^\circ = 100\cos 30^\circ \text{ N},$$

so that

$$Q = 89.7 \text{ N}.$$

Resolving along the perpendicular to the force of magnitude Q gives

$$P\cos 15^\circ = 100\cos 45^\circ \text{ N},$$

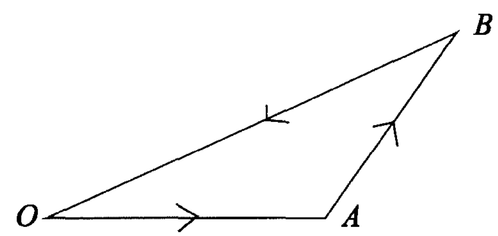
so that

$$P = 73.2 \text{ N}.$$

This method avoids the algebra of solving two equations but requires care in getting the right angles and some of you might prefer the algebra to the calculation of angles!

Triangle of forces

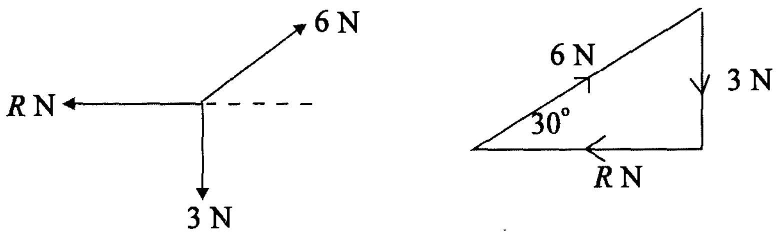
For equilibrium problems where there is only one unkown force an alternative method is to find the resultant of the known forces. The unknown force is then of the same magnitude as this resultant but opposite in direction. This method is considered in more detail in 2.3 but, for three forces acting at a point, it reduces to a geometric one. The triangle rule gives that the resultant of two forces represented in magnitude and direction by OA and AB is represented by OB. Therefore the third force necessary for equilibrium is represented by BO. Therefore if OA is drawn to represent one force, AB to represent the second force and BO to represent the third, a closed triangle is formed as shown below.



This will only be true if the three forces are in equilibrium, for any three arbitrary forces a closed triangle would not be formed. The triangle is known as the "triangle of forces", and in forming it you must draw the lines representing the forces "in order". This means that the arrows must follow each other and the arrow heads must not point towards each other. Several forces acting at a point form a closed polygon, the "polygon of forces" when they are in equilibrium but calculations for such polygons can be complicated. You are not yet able to solve problems involving calculating the sides and angles of a general triangle but you can use the triangle of forces when two forces are perpendicular to each other. (You can solve problems for triangles by scale drawing but you should not do this in a Mathematics examination unless you are specifically told that scale drawing is acceptable.)

The method will be illustrated by re-doing Example 2.7.

Example 2.7 (alternative solution)



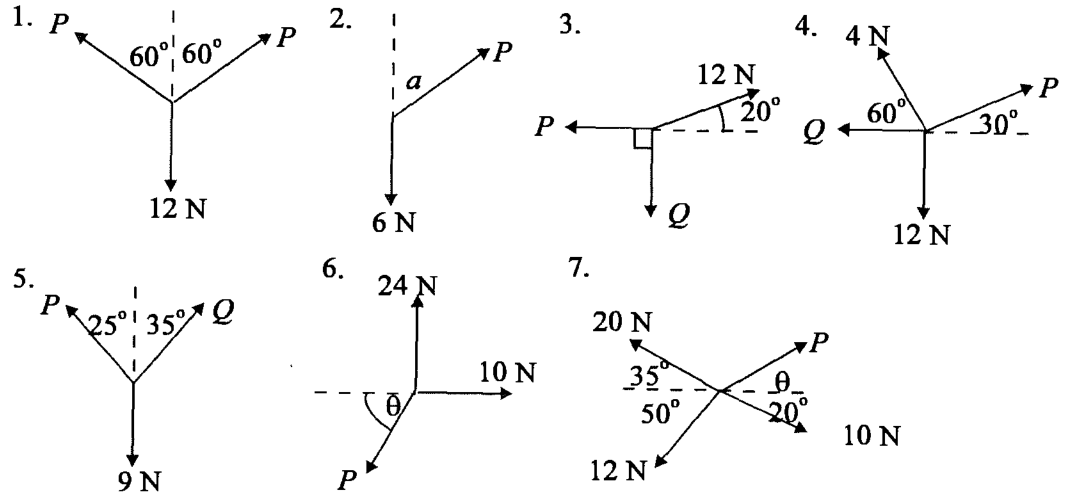
The forces acting are shown in the left hand diagram and the triangle of forces formed are shown in the right hand one.

From the triangle of forces

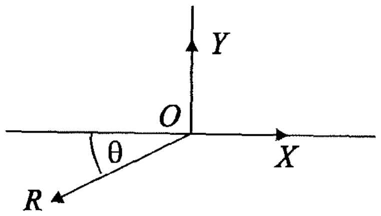
$$\cos 30^\circ = \frac{R}{6} \quad \text{i.e. } R = 6\cos 30^\circ = 3\sqrt{3}\,.$$

Exercises 2.3

Each of the problems 1 to 7 shows a system of forces in equilibrium. Find the unknowns in each case, giving forces to three significant figures and each angle to the nearest degree.



8 Three forces of magnitudes X, Y and R are shown in equilibrium at a point.



Find R and $\tan\theta$ when

(i) $X = 4$ N, $Y = 3$ N

(ii) $X = 7$ N, $Y = 2$ N.

2.2 Equilibrium problems involving physical modelling

All the problems in 2.1 have been straightforward because the various forces acting were prescribed exactly for you. In practical situations this will not be the case. You will have to use the data to model the problem and decide what forces are acting and then end up with the type of problems that you have been solving.

Modelling forces in the physical world

In setting up models of practical situations it is usually necessary to make simplifying assumptions about various objects and the forces exerted by, or on, them. These assumptions are often then summarised in a simple phrase or word (e.g particle, light string). It is very important that you realise precisely what assumptions are implied in using a particular description. The assumptions associated with the most common phrases used are summarised below in the form of a Glossary. This Glossary is intended to help you in interpreting questions that you have to answer. It should not be followed blindly. You should also read the more detailed explanation in 2.4 of the modelling assumptions that are summarised in the Glossary.
Some of the descriptions are a convenient fiction in that the situations described do not actually exist but, nevertheless, they often approximate to reality.

One major modelling assumption that you will be forced to make, since you have as yet only learnt methods for tackling problems where forces are acting at a point, is that any body that you consider has to be treated as a point. Therefore small objects like parcels and large ones like ships have to be modelled in the same way.

At a first reading you may not want to go into the reasoning in 2.4 behind the assumptions in 2.4 but you should look at the Glossary where the assumptions corresponding to various phrases are summarised.

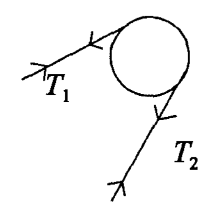
Glossary:

Force of gravity: Normally assumed to be constant and acting in the downwards vertical direction, whose magnitude is the weight of the body.
The weight of a body of mass m kg is mg N where g is the magnitude, approximately 9.81, of the acceleration of gravity in ms^{-2}. In most of the calculations in this book the approximation 9.8 will be used.
For a general body, the force of gravity acts through the centre of gravity. The location of this is a geometrical property of the body and for a uniform straight rod it is at its midpoint.

A simple way of measuring the weight of a body is by using bathroom scales. The reading on the scales is actually the reaction of the scales on the body. When the scales are stationary this is equal to the weight of the body.

Light A light body is one with zero weight or, more realistically, one whose weight is negligible compared to the other forces acting.

Strings are represented by thin lines and used to model ropes and even chains. When taut they exert a force, the tension, inwards from their ends, on bodies attached at the ends.
The tension in a string is effectively the pulling force exerted by one part of a string on the other part and can vary over the length of the string.
If a string is passed round any body such as a peg (or pulley), as shown in the diagram, then



the forces exerted on the peg are the tensions T_1 and T_2, acting at the points of tangency as shown.
Strings cannot exert a push away from their ends, nor exert any force perpendicular to themselves. There is no tension in a slack string.
If your calculations result in a negative or zero tension then you will have made a mistake.

Light strings The tension is constant throughout the length of a light string and, if taut, the string will be straight.

Inextensible strings, this means that the length does not change when a force is applied to the ends but the more realistic view is that the length does not change sufficiently for the tension to change.

Extensible (elastic) strings The length of such strings can vary and the tension will depend on the length.

The normal modelling assumption is that the tension (T) is directly proportional to the extension (x). This is **Hooke's law**. Symbolically Hooke's law states that

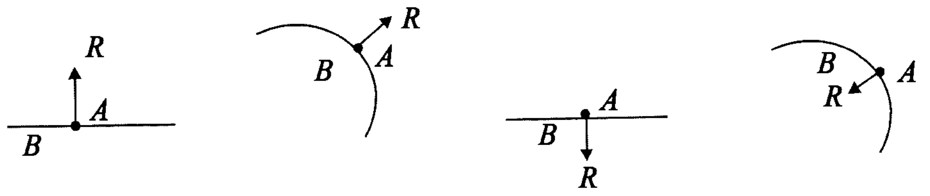
$$T = \frac{\lambda x}{l},$$

where λ is known as the modulus of elasticity of the string and l is the unstretched (natural) length of the string. The ratio $\frac{\lambda}{l}$ is called the stiffness of a string, though it is more often used in the context of springs rather than strings.

Springs have all the properties of elastic strings but can be compressed as well as extended and in this case exert a force (thrust) away from their ends. This thrust still satisfies Hooke's law with x now denoting the compression.

Thin rods (or beams) have all the properties of a string but can exert either a thrust or a tension at their ends. They are assumed to be rigid and, unlike a string, can sustain a force perpendicular to their lengths.

Smooth surfaces exert a reaction perpendicular to, and away from, themselves as shown in the two left hand diagrams below. The reaction of a smooth surface on a body cannot be negative. A zero reaction means that contact is about to be broken.

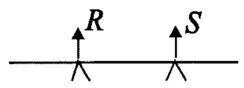


If there are two smooth bodies A and B in contact, then the reaction of A on B is equal in magnitude and opposite in direction to that of B on A, as shown in the two right hand diagrams above.
This is **Newton's third law.**

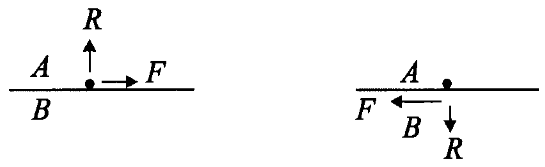
Smooth pegs The reaction at such pegs is normal to the peg and if a string is passed over such a peg, as shown in the diagram under Strings, then the tensions T_1 and T_2 are equal.

Smooth pulleys have, as far as Statical problems are concerned, the same properties as smooth pegs.

Simply (smoothly) supported rods (beams) The reactions of the supports on the rod or beam are perpendicular to it as shown below.



Rough surfaces The reaction on a body in contact with such surfaces is not normally perpendicular to the surface.
As well as the perpendicular (or normal) component of reaction R there is a force along the surface, the friction force of magnitude F, as shown in the left hand diagram below.



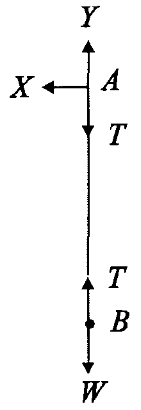
This force acts so as to oppose a tendency to move so that in the above diagram the motion of A would be to the left and $F \leq \mu R$, where μ is the coefficient of friction.
If a body is on the point of sliding on another, then the friction is said to be limiting and $F = \mu R$.
Newton's third law also holds for rough surfaces, as shown in the right hand diagram above.

In solving problems, the first step is to use the Glossary to interpret the problem and make the correct modelling assumptions to determine the type of force acting at each point. For strings, it is particularly important to mark the tensions at both ends of each straight part of each string. It should be remembered that, for a light string, the tension is constant throughout a given string but need not be the same in two strings tied to the same point.
All the forces acting on all bodies should be marked clearly on a **force diagram**. Making a clear force diagram is an essential first step in any problem solving. In all previous cases you have been presented with force diagrams but now you have to make your own.

If there are two bodies A and B in contact, then there will be a force diagram for each, and in making the force diagram you have to use Newton's third law which states that the force of A on B is equal and opposite to that of B on A.
The problems can then be solved as before by resolving in two different directions. You should remember that a good choice of directions in which to resolve can simplify your calculations as you saw in Example 2.9.
The problems that you can solve are those with forces acting at a point. There may be several points connected in some way and you will have to look at each point separately.
It is easier to start with problems not involving friction.

Problems not involving friction

<u>**Example 2.10**</u>
A particle of weight W is suspended in equilibrium from the end B of a light inextensible string AB. Find the tension in the string and the force necessary to hold the string at A.



Since the string is light, the tension will be constant throughout it and, as shown in the force diagram, will be acting inwards from the ends. There will be the force due to gravity of magnitude W acting vertically downwards at B. Nothing is known about the force at A and therefore it is safest to assume that it has a vertical component Y and a horizontal component X. All these forces are shown in the force diagram.

The forces acting at all points are in equilibrium. The only forces acting at B are the tension acting directly upwards and the weight acting directly downwards and therefore

$$T = W.$$

Looking next at A, the only horizontal component of force is X and therefore for equilibrium $\qquad X = 0.$

The component of force acting vertically upwards is Y and that downwards is T and therefore for equilibrium

$$Y = T = W.$$

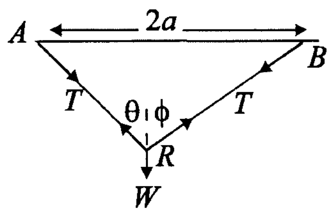
<u>**Example 2.11**</u>
A particle of mass m is suspended in equilibrium from the lower end B of a light elastic string AB, the upper end being held fixed. The elastic modulus of the string is $10mg$ and its unstretched length is a. Find, assuming that Hooke's law holds, the length of the string in the equilibrium position.
The forces acting are the same as in Example 2.10, and therefore the same diagram can be used, though, since the mass is given rather than the weight, W should be replaced by mg.
The equilibrium of the particle gives

$$T = mg$$

Since Hooke's law holds $\qquad T = \dfrac{10mgx}{a}$, where x is the extension.

Substituting $T = mg$ gives $x = \dfrac{a}{10}$ so that the total length of the string is $\dfrac{11a}{10}$.

<u>**Example 2.12**</u>
A small smooth ring R of weight W is threaded on a light inextensible string of length $8a$. The ends of the string are attached to two points A and B in a horizontal line and at a distance $2a$ apart. The system is in equilibrium in a vertical plane. Find the tension in the string.



The string is light and therefore the tension will be constant on both the straight parts of the string. The ring is smooth so that the tension is the same on both parts of the ring and therefore the forces acting are as shown in the above force diagram.
There is no reason to assume immediately that both strings are inclined at the same angle to the horizontal, though symmetry suggests this, and therefore it is safer to assume that the angle between the string and the vertical is θ to the left of the string and ϕ to the right.

Resolving horizontally the forces on the ring gives

$$T\sin\theta = T\sin\phi,$$

so that $\theta = \phi$ and therefore $AR = BR = 4a$ and R is directly below the middle point of AB.

Resolving vertically the components of the forces at the ring gives

$$2T\cos\theta = W.$$

The depth of R below AB is, by Pythagoras' Theorem, $\sqrt{15}a$ so that $\cos\theta = \dfrac{\sqrt{15}}{4}$

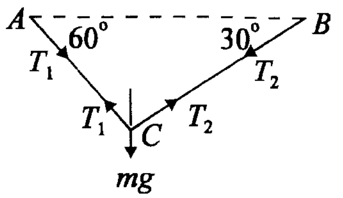
Therefore,

$$T = \frac{2W}{\sqrt{15}}$$

Example 2.13

Two light inextensible strings are each tied to a particle of mass m.

The other ends are attached to two points in a horizontal line so that the particle is in equilibrium with the strings inclined at angles of 30° and 60° to the horizontal. Find the tensions in the strings.



The force diagram is shown above. Since the strings are tied to the particle the tensions in them could be different and will be denoted by T_1 and T_2 respectively. In this case the mass of the particle is given rather than the weight so that the force due to gravity is mg.

There are two unknowns and they can be found by equating components in two different directions. The most obvious choice is to equate components vertically and horizontally, this would give two equations involving T_1 and T_2 and these then have to be solved for T_1 and T_2.

An alternative is to consider components along the two strings, these are perpendicular and therefore, for example, the tension along AC does not have a component along BC. The problem is very similar to that of Example 2.9.

The component of the force of gravity along AC is $mg\cos 30°$ so

$$T_1 = mg\cos 30° = \frac{\sqrt{3}mg}{2}$$

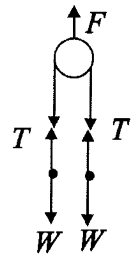
The component of the force of gravity along BC is $mg\sin 30°$ so

$$T_2 = mg\sin 30° = \frac{mg}{2}$$

Example 2.14

Two small particles, each of weight W, are attached to the ends of a light inextensible string. The string passes over a small smooth peg and the particles are in equilibrium in a vertical plane. The string, at the points where it loses contact with the peg, is vertical. Find the force exerted on the peg.

Since the peg is smooth the tension on both sides of the peg is the same and acts vertically down and the force diagram is :-



The strings therefore do not exert a horizontal force on the peg .

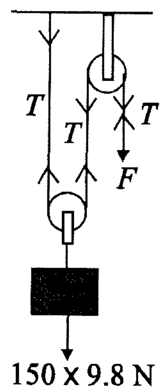
Resolving vertically for either particle gives

$$T = W.$$

The only force exerted on the peg by the string is $2T = 2W$ acting downwards.

Example 2.15

The pulley system shown below is used to support a crate of mass 150 kg. Find, assuming the pulleys are smooth and light, and the rope is light and modelling the crate as a particle, the force that has to be applied at the end of the rope to maintain equilibrium.



The forces acting are shown in the diagram. Since the pulleys are smooth, the tensions in all parts of the rope are the same. The equilibium of the crate gives

$$2T = 150 \times 9.8 \text{ N},$$

so that $T = 735$ N. Equilibrium at the end of the rope gives

$$F = T,$$

so that the required force is 735 N.

Exercises 2.4

In numerical examples, forces should be found to three significant figures and g should be taken as 9.8 ms^{-2}.

1 A small body is suspended in equilibrium from a fixed point by a light inextensible string. Find

(a) given that the mass of the body is 0.4 kg, its weight and the tension in the string,

(b) given that the weight of the body is 14.7 N, its mass and the tension in the string.

2 A small parcel is placed on a horizontal table. Modelling the parcel as a particle, find

(a) the reaction perpendicular to the table when the particle is of mass 3 kg,

(b) the mass of the parcel given that the normal reaction is 19.6 N.

Questions 3 to 5 refer to a body of weight W N suspended from a fixed point by a light elastic string of natural length a m, elastic modulus λ N. The extension is denoted by x m.

3 Find x, given $W = 21$, $a = 2$, $\lambda = 105$.

4 Find λ, given $W = 50$, $x = 0.2$, $a = 4$.

5 Find a, given $W = 30$, $x = 0.2$, $\lambda = 210$.

Questions 6 and 7 refer to a particle of mass 0.4 kg suspended by a light inextensible string, the other end of which is attached to a fixed point.

6 The particle is acted on by a horizontal force so that it is in equilibrium with the string inclined at an angle of 40° to the downward vertical. Find the force.

7 The particle is maintained in equilibrium with the string inclined at an angle of 30° to the downward vertical by a force acting on the particle perpendicular to the string. Find the force and the tension in the string.

8 A particle of mass 0.3 kg is suspended by two light inextensible strings from two fixed points on the same horizontal level. The strings are inclined at angles 25° and 35°, respectively, to the horizontal. Find the tensions in the strings.

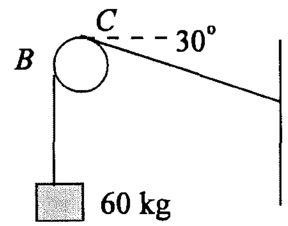
9 A smooth ring R of mass m slides on a light inextensible string whose ends A and B are fixed at two points on the same level. A horizontal force of magnitude P is applied at R

so that the ring is in equilibrium vertically below A, with BR inclined at an angle α to the vertical. Find P.

10 Two identical light elastic strings AB and BC, each of natural length 0.8 m and modulus 3000 N, are joined together at B. A particle of weight 900 N is attached to C and is suspended in equilibrium by the composite string ABC with the end A fixed. Find the length ABC.

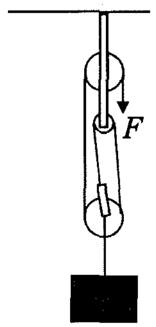
11 Assume now that in question 10 the strings are not joined to each other at B but both attached to a particle of weight 20 N with the particle being between the strings. Find the length ABC when the particle of weight 900 N is now suspended in equilibrium.

12



The diagram shows a crate of mass 60 kg supported by a rope passing over a small pulley. The other end of the rope is attached to a fixed point. The pulley is circular and the rope just loses contact with the peg at B and C. The rope is vertical at B and at an angle of 30° to the horizontal at C. Neglecting the weight of the rope, modelling the crate as a particle and assuming the pulley to be smooth, find the tension in the rope and the horizontal and vertical components of the force acting on the pulley.

13

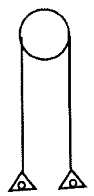


Find the force, F, that has to be applied to the end of the rope in order that the pulley system can support the box of weight 80 kg. The pulleys are smooth and their mass, and that of the ropes, may be neglected. Also all parts of the string may be assumed to be vertical.

14 A light elastic string AB, of natural length 1.5 m and modulus 200 N, has the end A fixed and a heavy particle attached to the end B. A horizontal force of magnitude F is then applied at B so that the system is in equilibrium with AB taut and inclined at an angle of 30° to the downward vertical, and $AB = 1.8$ m. Find the value of F and the mass of the particle.

15 Two light strings, attached to a particle P of mass M, pass over two smooth pegs at the same level and hang vertically in equilibrium with masses $3m$ and $4m$ at their ends. Given that the strings at P are perpendicular to each other, find the ratio $\frac{m}{M}$.

16



The diagram shows a light string, with a scale pan attached at each end, passing over a small peg. The mass of each scale pan is 0.1 kg and the system is initially in equilibrium with 0.9 kg in each scale pan and the string at the points where it just loses contact with the peg is vertical. Find the tension in the string.

It was found that a further 0.4 kg could be placed in one scale pan before equilibrium was broken. Find the tensions in the two parts of the string when equilibrium is about to be broken.

It may be assumed that motion did not take place immediately when the mass in one scale pan was increased because the peg is rough. A model taking into account the roughness of the peg shows that when the string is about to move, the tension increases along the string in the direction in which motion would occur so that the ratio of the tensions at the two points at which the string is about to lose contact with the peg is $e^{\mu\pi}$, where μ is the coefficient of friction. Use this model to estimate the coefficient of friction.

Problems involving friction

The simplest types of problems involving friction are those when the frictional force necessary for equilibrium has to be found and these are essentially the same as the problems you have already solved.

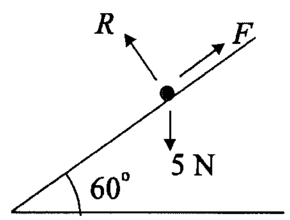
The next class, in order of difficulty, is that when equilibrium is on the point of being broken and in these cases the equilibrium is limiting so that, provided the direction of friction is known, a force diagram can be drawn which includes the frictional forces. The problem can then be solved by resolving in two different directions.

In all other cases the best tactic is to form a force diagram showing arbitrary values of the friction force F and the normal reaction R. Expressions for F and R can be obtained by resolving in the usual way and then the condition $F \le \mu R$ applied. The only snag is that you may not always have chosen the correct direction for friction and "your" F may not be the magnitude. If this happens then you will be probably end up with something which is automatically true or nonsense and no progress will have been made. You can avoid this by using $-\mu R \le F \le \mu R$. It is particularly important that in questions where an inequality is required that you use $F \le \mu R$ (or $-\mu R \le F \le \mu R$) rather than assume friction is limiting and then put in an inequality on the last line. If the inequality is not given then there is a 50% chance of you getting the inequality the "wrong way"!

__Example 2.16__

A particle of weight 5 N is in equilibrium on a rough plane inclined at an angle of 60° to the horizontal. Find the normal reaction and the friction force acting on the particle.

The force diagram is :



For problems involving inclined planes it is often easier to take components along and perpendicular to the plane rather than horizontally and vertically. The components of the weight are 5 sin 60° N down the plane and 5 cos 60° N perpendicular to the plane and in the sense away from the particle into the plane.

Therefore resolving along and perpendicular to the plane gives

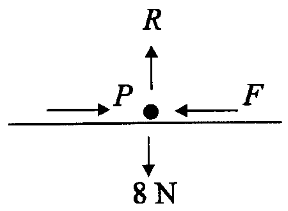
$R = 5\cos 60°\text{ N} = 2.5\text{ N}$ and $F = 5\sin 60°\text{ N} = 2.5\sqrt{3}\text{ N}$.

The direction of F was taken to be up the plane since any motion would be downwards. If the wrong direction had been chosen then this would have shown up in a negative value of F.

__Example 2.17__

A particle of weight 8 N is at rest on a rough horizontal table, the coefficient of friction between the particle and the table is 0.4. When a horizontal force of magnitude P is applied to the particle it is just about to slide. Find the value of P.

The force diagram is:



Resolving vertically gives $\qquad R = 8$ N.

In this case the particle is about to move so that friction is limiting and therefore

$$F = \mu R = 3.2 \text{ N}.$$

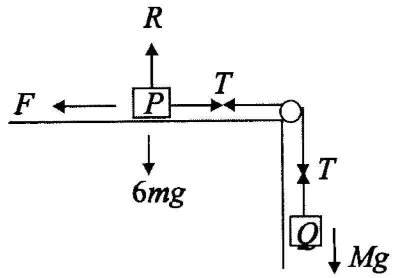
Resolving horizontally gives $\qquad P = F = 3.2$ N.

Example 2.18

A particle P of mass 6 m lies on a rough horizontal table, with coefficient of friction $\frac{1}{4}$.

The particle is attached by a light inextensible string to a second particle Q. The string passes over a smooth pulley at the edge of the table and Q hangs in equilibrium. Find the greatest possible mass of Q.

The force diagram is :



It is assumed that Q has mass M. If motion were to occur it would be to the right and therefore the friction force is to the left. The only forces acting on Q are vertical and, resolving vertically for Q gives

$$T = Mg.$$

Resolving horizontally for P gives

$$F = T,$$

so that

$$F = Mg.$$

Resolving vertically for P gives

$$R = 6mg.$$

Therefore

$$\frac{F}{R} = \frac{M}{6m}$$

The maximum value of this ratio is $\frac{1}{4}$ so that $\frac{M}{6m} \le \frac{1}{4}$

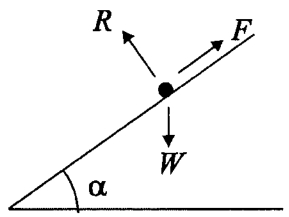
So the greatest possible mass of Q is $\frac{3m}{2}$

Example 2.19

A heavy particle of weight W is placed on a rough plane inclined at an angle α to the horizontal. The coefficient of friction between the plane and the particle is μ. Show that equilibrium is not possible unless $\tan \alpha \le \mu$.

Find, when $\tan \alpha > \mu$, the least value of the magnitude of a force acting up the line of greatest slope of the plane which will maintain equilibrium.

The force diagram for the first part is:



Since the particle is likely to slip down the plane F will act up the plane and resolving along and perpendicular to the plane as in Example 2.16 gives

$$F = W \sin \alpha, \quad R = W \cos \alpha,$$

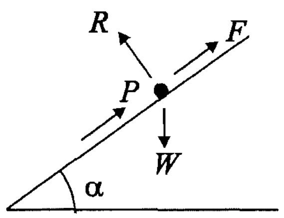
and

$$\frac{F}{R} = \tan \alpha.$$

Therefore $\tan \alpha \le \mu$ for equilibrium.

In the second part the force of magnitude P acts up the plane. The solution to the first part shows that for $\tan \alpha > \mu$ the particle would slip down the plane, therefore the least force to obtain equilibrium is that which just stops the particle sliding down. The friction force will still act up the plane.

The force diagram is therefore:



Resolving along and parallel to the plane gives

$$F + P = W \sin \alpha,\ R = W \cos \alpha.$$

The condition $F \le \mu R$ gives

$$W \sin \alpha - P \le \mu W \cos \alpha$$

and therefore $\quad W \sin \alpha - \mu W \cos \alpha \le P.$

Therefore the least value of P is $W(\sin \alpha - \mu \cos \alpha)$.

It is a good idea in examples like this involving friction to check that the answer is sensible. Here it is in the sense that, since $\tan \alpha > \mu$, it is at least positive!

Example 2.20

Find for the problem in Example 2.19, when $\tan \alpha > \mu$, the greatest value of the magnitude of a force acting up the line of greatest slope of the plane which will maintain equilibrium.

The greatest value of P would be when the particle is about to slip up the plane and F is acting downwards. If P is assumed to still act in the same direction then the same equations as before are found. Applying the condition $-\mu R \le F \le \mu R$ gives

$$\mu W \cos \alpha \le W \sin\alpha - P \le \mu W \cos \alpha,$$

so that $\quad W \sin \alpha - \mu W \cos \alpha \le P \le W \sin \alpha + \mu W \cos \alpha.$

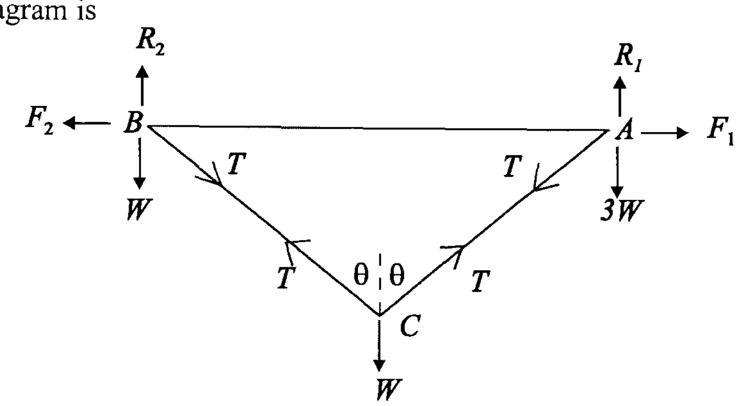
The left hand inequality gives the least value of P and the right hand one the maximum value. The maximum value occurs when the particle is about to move up the plane so friction is acting downwards and the force has to overcome both friction and the force of gravity.

Example 2.21

Two small rough rings A and B, of weights $3W$ and W respectively, slide on a fixed, rough, horizontal rod. The coefficient of friction between the rod and each ring is 0.5. A light inextensible string is threaded through a smooth ring of weight W and its ends are attached to A and B. The whole rests in equilibrium in a vertical plane. Explain why

both parts of the string are inclined at the same angle θ to the vertical and find the greatest value of θ for which equilibrium is possible.

The force diagram is



If the strings were inclined at different angles θ and ϕ then resolving horizontally at C would give

$$T \sin \theta = T \sin \phi,$$

so that

$$\theta = \phi.$$

This is a fairly complicated problem in that it is necessary to resolve at A, B and C.

Resolving vertically at C gives

$$2T \cos \theta = W.$$

Resolving horizontally and vertically at A gives

$$T \sin \theta = F_1,\ R_1 = 3W + T \cos \theta.$$

Resolving horizontally and vertically at B gives

$$T \sin \theta = F_2,\ R_2 = W + T \cos \theta.$$

Substituting for $T \cos \theta$ gives

$$R_1 = \frac{7W}{2},\ R_2 = \frac{3W}{2} \text{ and } F_1 = F_2 = \frac{W\tan\theta}{2}.$$

The friction forces are the same at both A and B and since $R_2 < R_1$ it follows that slipping will first occur at B when

$$\frac{F_2}{R_2} = \frac{1}{3} \tan \theta \le 0.5.$$

The maximum value of $\tan \theta$ is 1.5 giving the maximum value of θ as approximately 56.3°.

Exercises 2.5

In numerical examples, g should be taken as 9.8 ms^{-2} and answers given correct to three significant figures.

1 A particle is in equilibrium on a rough horizontal table. A string is attached to the particle and is inclined at an angle of 40° to the horizontal. T denotes the tension in the string and F the friction force. Find

(a) F given that $T = 40$ N, (b) T given that $F = 60$ N.

2 A particle of mass 1.5 kg, at rest on a rough horizontal plane, can just be moved by a horizontal force of magnitude 5 N. Find the coefficient of friction.

3 A particle of mass 2 kg is in equilibrium on a rough horizontal plane, the coefficient of friction being 0.4. Find the least force which, acting (i) horizontally, (ii) at an angle of 30° to the upward vertical, would just move the body along the plane.

4 A particle of mass 3 kg is placed on a rough plane inclined at an angle of 50° to the horizontal. The coefficient of friction between the plane and the particle is 0.25. Find the least force acting along the line of greatest slope of the plane required (i) to prevent the particle from sliding down, (ii) to move it up the plane.

5 A particle of weight 80 N is held in limiting equilibrium on a plane inclined at an angle of 30° to the horizontal by a horizontal force. Given that the coefficient of friction is 0.4 find the magnitude of the force when

(i) the particle is about to slip up the plane,

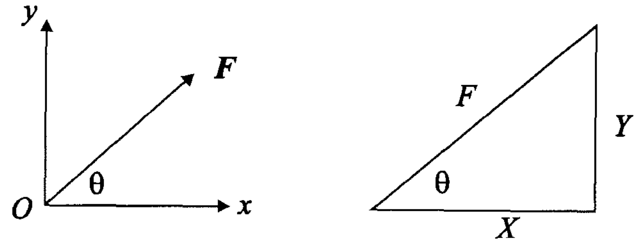
(ii) the particle is about to slip down the plane.

6 A particle of mass 3 kg is on the point of sliding down a rough plane inclined at an angle α to the horizontal, when a force of magnitude 5 N is applied up the plane along a line of greatest slope. When the force is increased to 10 N the particle is on the point of moving up the plane. Find $\sin\alpha$.

2.3 Calculation of the resultant of forces acting at a point

The main purpose of calculating the resultant is for use in problems involving motion but the idea of a resultant is also handy in equilibrium problems. For example, if there are two forces acting and a third has to be included to produce equilibrium then the third force would have the same magnitude as the resultant of the other two but act in the opposite direction. This could occur with wires at the top of a telegraph pole when two had been attached and the third had to be placed so as to give equilibrium.

The components of the resultant of several forces acting at a point are the sum of the components of the individual forces. Therefore if the resultant of a force can be calculated from its components then the resultant of any number of forces can be found.



The components of a force F inclined at an angle θ to the x direction as in the diagram are

$$X = F\cos\theta,\quad Y = F\sin\theta.$$

If θ is acute then F is the hypotenuse of a right angled triangle whose other sides are X and Y and therefore by Pythagoras' theorem

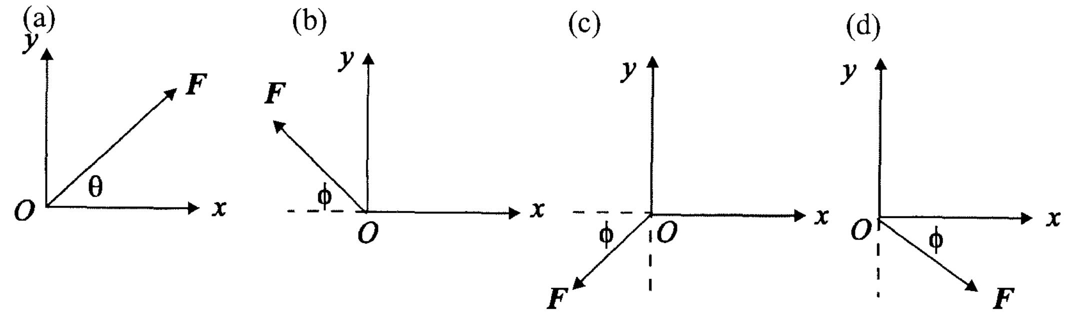
$$F = \sqrt{X^2 + Y^2}.$$

It is actually possible to prove that this is true even if θ is not acute (so that one or both of X and Y may be negative) and from now on it will be assumed to be true for all X and Y. Therefore the magnitude of the resultant can be calculated easily.

Finding the direction is a bit trickier. Dividing the components gives

$$\tan\theta = \frac{Y}{X}.$$

You have met this kind of equation before in working with right angled triangles. When X and Y are both positive, cos θ and sin θ are also positive so that θ is acute and you can find it by using the $\tan^{-1}$ function on your calculator. When one of X or Y is negative it is necessary to be more careful and the first step is to find out in which quadrant the line representing the resultant lies. The four possibilities are shown below.



The acute angle ϕ between the resultant and the x axis can be found from

$$\tan\phi = \frac{|Y|}{|X|},$$

i.e. you drop the minuses on the components. This lets you work out the exact position of the resultant and you can then work out the angle with the positive x-direction or any other line.

It is possible to carry out the calculation more directly by finding F first and using the $\cos^{-1}$ or $\sin^{-1}$ functions to find θ. You still have to determine the quadrant in which θ lies and use the symmetries of the trigonometric functions (for θ in degrees, these are $\sin(180-\theta) = \sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(180+\theta) = \tan\theta$). Until you have had more practice in trigonometry you may find this second method a bit tricky.

In working out resultants you can choose any two perpendicular reference directions that you like, but for two forces you could simplify things by taking one reference direction parallel to one of the forces.

Example 2.22

Find the resultant when (a) $X = 5$ N, $Y = 2$ N, (b) $X = -4$ N, $Y = 4$ N (c) $X = -6$ N, $Y = -3$ N, (d) $X = 3$ N, $Y = -1$ N.

(a) This is the simplest case, corresponding to diagram (a) above, so that

$$R = \sqrt{5^2 + 2^2}\ \text{N} = 5.39\ \text{N, also } \tan\theta = \frac{2}{5} \text{ so that } \theta = 21.8°.$$

(b) This corresponds to diagram (b) above where the line representing the resultant lies in the second quadrant.

The resultant is therefore $\sqrt{4^2 + 4^2}$ N = 5.67 N, $\tan\phi = \frac{4}{4}$ so $\phi = 45°$ and therefore the resultant is at an angle of 135° to the positive x-direction.

(c) This corresponds to diagram (c) above with the line representing the resultant lying in the third quadrant.

The resultant is therefore $\sqrt{6^2 + 3^2}$ N = 6.71 N, $\tan\phi = \frac{1}{2}$ so $\phi = 26.6°$ and therefore the resultant is at an angle of 206.6° to the positive x-direction.

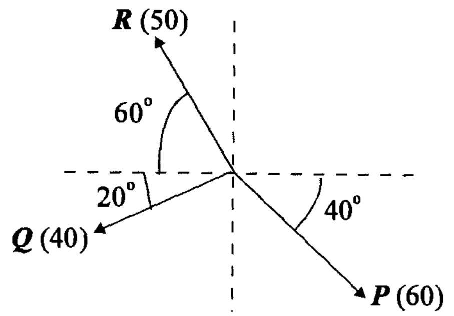
(d) This corresponds to diagram (d) with the line representing the resultant being in the fourth quadrant.

The resultant is therefore $\sqrt{3^2 + 1^2}$ N = 3.16 N, $\tan\phi = \frac{1}{3}$ so $\phi = 18.4°$ and therefore the resultant is at an angle of $-18.4°$ (or 341.6°) to the x-direction.

When finding the resultant for several forces acting,there will be an additional step of adding the separate components to get the components of the resultant force.

Example 2.23

The diagram shows three forces P, Q and R acting at a point. The magnitudes (in newtons) are shown in brackets. Find the magnitude and direction of their resultant.



The components of P, Q and R, respectively, to the right across the page are (in newtons) 60 cos 40°, – 40 cos 20° and – 50 cos 60°. The component of the resultant is the sum of these which is –16.6 N.

The components of P, Q and R, respectively, up the page are (in newtons) – 60 sin 40°, – 40 sin 20° and 50 sin 60°. The component of the resultant is the sum of these which is – 8.95 N.

The resultant is of magnitude $\sqrt{16.6^2 + 8.95^2}$ N = 18.9 N, the line representing the resultant lies in the third quadrant. The tangent of the acute angle between the direction of the resultant and that of the dotted line is $\frac{8.95}{16.6} = 0.539$, so this angle is 28.3°. Therefore the resultant makes an angle of 208.3° with the line to the right and across the page.

Example 2.24

The three forces in the previous example model the forces in three horizontal wires at the top of a telegraph pole. Find the position of, and tension in, a fourth horizontal wire to be placed so that the four forces will be in equilibrium.

The fourth wire will be in the opposite direction to the resultant of the other three, i.e. it acts at an angle of 28.3° to the line to the right and across the page and the tension in this wire is 18.9 N.

Exercises 2.6

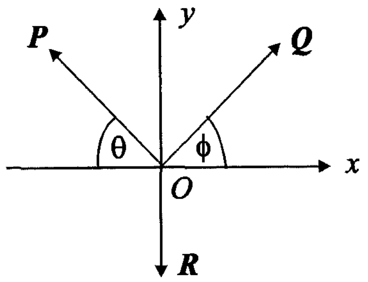
In numerical examples answers should be given correct to three significant figures.

1 The x- and y- components of a force are denoted by X and Y. Find the magnitude and direction, referred to the positive x-direction, of the resultant when

(a) $X = 7$ N, $Y = 3$ N, (b) $X = 4$ N, $Y = -8$ N, (c) $X = -3$N, $Y = 11$ N,

(d) $X=-5$ N, $Y=-13$ N, (e) $X=-7$ N, $Y=4$ N, (f) $X=-3$ N, $Y=-3$ N.

2 The diagram shows three forces $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ acting at a point O. Find the magnitude of their resultant, and the direction it makes with the positive x- direction, when



(a) $P=1$ N, $Q=2$ N, $R=5$ N, $\theta=20°$, $\phi=30°$,
(b) $P=8$ N, $Q=3$ N, $R=4$ N, $\theta=40°$, $\phi=60°$,
(c) $P=6$ N, $Q=6$ N, $R=1$ N, $\theta=50°$, $\phi=20°$.

3 Find the additional force that will have to be introduced into each of the cases in the previous exercise in order that the system of four forces is in equilibrium.

2.4 Modelling assumptions

Force of gravity

You know that if you release anything just above the surface of the earth then it will drop down. There is therefore a force acting on it. This is the force of gravity acting on the body. It acts towards the centre of the earth and its magnitude is the weight of the body.

The weight of the body can also be expressed in terms of the mass, m, of the body and g, the acceleration due to gravity. These quantities will be defined for you more precisely later but for the time being all you need to know is that, for any body, there is a precisely defined quantity called its mass and g is approximately 9.81 ms^{-2}. The unit of mass is the kilogram (abbreviation kg). The weight in newtons of a body of mass m kg is mg where g is measured in ms^{-2}.

The actual force exerted by gravity varies in magnitude with the distance from the earth's centre and also varies with latitude. In most circumstances these variations are ignored and the usual modelling assumption is that the weight of a particle is constant and that the force of gravity acts along the vertical.

For the simple model of a body as a particle, the force of gravity acts at the point occupied by the particle but for a general body, which is a collection of particles, the situation is not so clear. There is however, for any body, a unique point through which the force of gravity acts, this point is known as the centre of gravity of the body. You will not be expected to know the position of the centre of gravity of any particular

3-dimensional body. In the one-dimensional case, the centre of gravity of a thin uniform rod lies at its midpoint. For 2-dimensional problems, our attention is restricted to uniform plane figures. This is explained in greater detail in Chapter 7.2.

One device for measuring weight is the spring balance, another is the ordinary bathroom scales. The actual reading on the scales gives the magnitude of the reaction of the scales on the body and, when the scales are not moving, this is equal to the weight of the body.

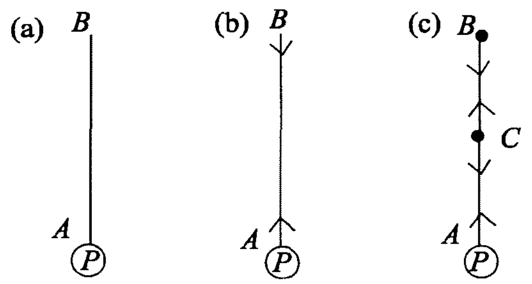
These devices do not actually measure the true weight (i.e. the magnitude of the force of gravity) since there should be a slight correction due to the effect of the rotation of the earth. This is normally neglected but that in itself is a modelling assumption.

A body without weight (more precisely one where the gravitational force acting on it may be neglected compared to other forces) is described as light.

You should be careful to notice that there is a difference between being without weight and being weightless, which is something you may have heard of. Weightlessness is normally associated with motion and a body is said to be weightless if there is no reaction between it and a surface it is in contact with. For example if someone were unlucky enough to be standing on bathroom scales in a lift which was falling freely under gravity then the scale would not show a reading. Similarly if you were falling freely and holding a suitcase then you would not feel the weight of the suitcase and so to you it would appear weightless (see Example 5.3). Even these statements are not precisely correct because they assume that there are no forces other than gravity acting, i.e. there are no resistances. This would mean that these statements would only be valid in a vacuum.

Strings

A small body P is attached to one end A of a string AB, the other end B is held fixed so that the body hangs at rest as shown in diagram (a) below.



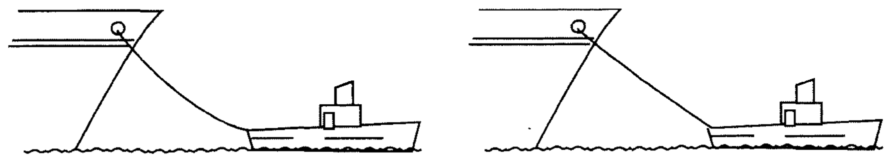
To simplify matters it will be assumed that P is modelled as a particle. There is the force of gravity acting vertically downwards on P and therefore, since it is at rest (i.e. in equilibrium) there is an equal and opposite force acting in the string from A to B. Such a force directed from one end to the other is called the tension. The individual holding the string at B would experience a force acting in the direction of B to A. Therefore, a string exerts a pull (tension) at its extremities as shown in diagram (b) above. For any point C between A and B, the part AC will exert some force on the part BC and this force will be in the direction CA and the part BC will exert a force on AC in the direction CB. The situation is as shown in diagram (c).

Therefore, a string sustains a tension at all points along its length and the tension is in fact the force of interaction between two parts of the string.

The word string in Mechanics implies something which has length but no cross section so that it can be modelled by a thin line or curve. A string can only pull from its ends and not push. It is also assumed that it is perfectly flexible i.e. it cannot exert a force perpendicular to its length.

If the string in the above diagrams is assumed light then there will be no force due to gravity acting at any point of the string and therefore the force exerted by the part BC will be equal in magnitude to that exerted by the part AC. Therefore the tension will be constant. It is possible to prove that assuming a string is light means that the tension will be constant throughout its length and that, if any two points of a light string are held fixed, the string in the region between them will be straight.

Cables and ropes of all kinds are modelled as light strings and this need not always be an adequate model. For example the left hand diagram below shows a tug just attached to a ship and the right hand one shows the tug about to move the ship.

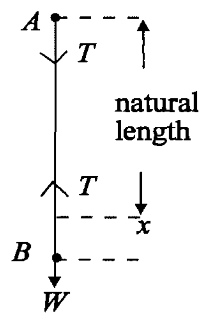


A light string is obviously not a good model in the left hand case since the rope is sagging but it might be reasonable in the second case. The difference is that in the second case there is considerable tension in the rope and that tension is considerably greater than the weight of the rope. Therefore modelling a rope by a light string is effectively ignoring the weight of the rope relative to the tension acting on it.

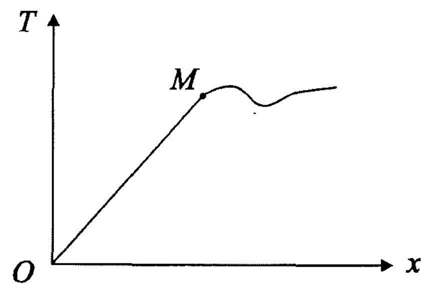
Elastic strings

You know that if you pull at the two ends of a piece of string of the type used to tie up parcels then, as far as you can see, the distance between the two ends will stay the same. If, on the other hand, you pulled the two ends of a thin piece of elastic then the distance

between the ends will increase. One way of distinguishing between these is to say that the parcel string is inextensible and that the piece of elastic is extensible. The difference is however more than this and modelling elastic strings is not always straightforward.



The diagram shows an elastic string AB suspended from a fixed point A and a weight W is attached to the end B. The length of AB can be found and the extension of the string determined. (Determination of the extension requires knowing the unstretched, or natural, length of the string. This can be found by putting the string so that it is just straight on a table and measuring its length). By putting different weights at B the extensions corresponding to different weights can be found. Since the weights are in equilibrium the tension in the string is equal to the weight. Therefore the tensions corresponding to different extensions can be found. Experiments of this kind have been carried out for many materials and the graph of the tension T against extension x is roughly as shown below.



Up to a certain value of T, the graph will be a straight line OM, thereafter it may take on a more complicated form, one possibility is shown in the above diagram. For sufficiently large tensions the string eventually breaks.

For tensions corresponding to the part OM of the graph, if the weights are removed then the string regains its natural length and if the experiment is repeated then the line OM will again be obtained. If the weights are removed for tensions greater than those corresponding to the part OM then the string will not regain its natural length and the same graph will not be obtained on repeating the experiment. The tension corresponding to the point M represents the elastic limit of the string.

A light extensible (elastic) string possesses some of the properties of a light inextensible string in that it can only sustain a tension (that is, it cannot push) and the tension is constant along its length. It differs in that, as the adjective extensible suggests, the length of the string is not constant.

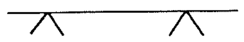
The usual model of an elastic (extensible string) is that it possesses all the attributes of a string but that the tension is described by the line OM i.e. the tension is directly proportional to the extension. This is Hooke's law which states that when an elastic string of natural length l is extended by an amount x then the magnitude of the tension T in the string is given by $$T = \frac{\lambda x}{l}$$

where λ is an experimentally determinable constant known as the modulus of elasticity of the particular string, while λ/l is often called the stiffness and also the string constant.

Springs A spring is effectively a spiral of thin wire and it is usually assumed that it can be modelled as an elastic string though it has one additional property in that it can sustain a thrust as well as a tension. This means that a spring can be compressed and when compressed there is a thrust acting outwards at the ends.

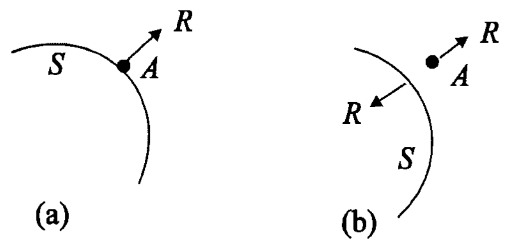
Hooke's law still applies to a spring, though, for a compression, the tension becomes a thrust.

Rods A rod, like a string, is modelled as a line. Rods are assumed to be rigid and inflexible so that they are modelled by straight lines. There are two assumptions made, using a rod model, which are not valid for a string model. One is that a rod can exert thrusts as well as tensions, the other is that a rod can sustain forces perpendicular to itself. This means that a rod can be supported by forces applied at two points of itself as shown in the diagram and can be used to give, for example, a simple model of a plank across a river or even of a bridge. In reality most planks, if their weight cannot be neglected, will bend slightly if supported as above and are slightly elastic. The rod model ignores this possibility.



Smooth surfaces

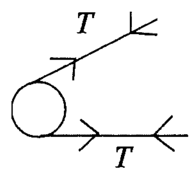
If a small body A is in contact with a surface S as shown below in diagram (a)



there will be some force exerted by S on A. A smooth surface is defined to be such that its reaction on A is normal to S at the point of contact and in the direction from S to A. There will also be, by Newton's third law, an equal and opposite force acting on A due to S and in some problems it is necessary to show the bodies as slightly separated as shown in diagram (b) above and the interactive forces on each body shown clearly. The reaction of the surface is always away from it and, if it becomes zero (or is inwards) in your calculations, then this means that contact is about to be (or has been) lost.

If something like a book were pushed along a perfectly smooth table then only the slightest push would be necessary to move it and it would then continue to move without further effort. This is because a smooth surface would not exert a force tangential to itself. In reality, no surface is perfectly smooth. If one did exist then it would not be possible to walk or drive on it. This is because the act of walking or driving exerts a tangential force on the road and, by Newton's third law, the road exerts an equal and opposite force on the foot or wheel and this is what makes motion possible.

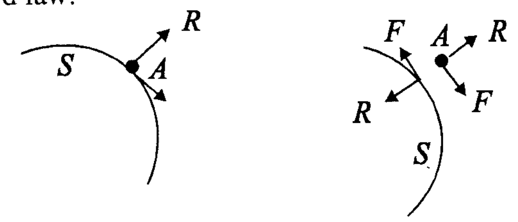
Smooth peg (or pulley) A peg or a pulley is used as shown below to change the



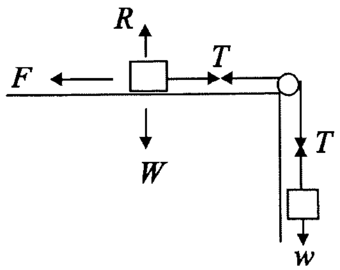
direction of a rope (modelled usually as a string). A smooth peg is one such that the only force that it exerts is perpendicular to its surface. Also if a string is passed round a smooth peg as shown above, then the tension on both sides of the string is the same. In Statics a smooth pulley is modelled as a smooth peg. The modelling of a pulley, when motion is involved, is more complicated and is discussed in 5.3.

Rough surfaces

As mentioned above, surfaces, in reality, exert a tangential force on anything in contact with them. If you push a book on a table then initially you will experience some resistance, then the book suddenly slips and, if you are sufficently perceptive, you might notice that less effort is required to keep the book moving than was necessary to start it. Any surface which exerts a tangential (i.e sideways) force is said to be rough. The force at the point of contact with a rough surface has two components, the reaction R perpendicular to the surface and a component F, the friction force. This is illustrated in the following diagram where the forces on both bodies in contact are shown, using Newton's third law.



The following diagram illustrates an experiment that can be used to determine the behaviour at a point of contact with a rough surface.



A small block of weight W is placed on a table and a string attached to the block passes over a smooth pulley at the edge of the table and a weight w is attached to the free end of the string so that the system is in equilibrium. The block is modelled as a particle, the only forces acting on it are its weight W vertically downwards, the normal reaction R acting vertically upwards, the tension T in the string and the force of friction F which is shown acting to the left. Resolving horizontally and vertically for the block gives

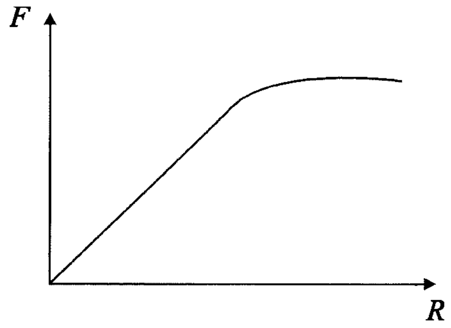
$$R = W, F = T,$$

and resolving vertically for the hanging weight gives

$$T = w.$$

The last equation assumes the pulley is smooth and that the string is light, it also shows since tension is positive that the correct direction was chosen for the force of friction. Initially the values of w can be increased, for given W, without disturbing equilibrium. At a particular value of w the block just slips, and the values of R (= W) and F (= w) at

slipping recorded. The process can be repeated for various values of W and a graph of R against F (at slipping) drawn. The graph will be found to be of the form below, being initially a straight line and then curving.



On the straight line part

$$F = \mu R,$$

and μ is called the coefficient of friction, or, more correctly, the coefficient of static friction and is a property of the two surfaces in contact. The normal model ignores the curved part of the curve and it will always be assumed that the relation between the normal reaction and the friction force at slipping is linear.

From experiments, the following summarise the properties of the friction force at a rough surface.

(a) The force of friction acts in the sense so as to prevent motion of a body.

(b) Until the magnitude of the force of friction reaches a limiting value its magnitude is just sufficient to prevent slipping.

(c) At the limiting value, $F = \mu R$ where μ is called the coefficient of friction.

(d) Until slipping occurs $F \leq \mu R$.

In many problems F is used to denote the component in a particular direction and this may not always be in the direction in which the friction force is acting, therefore F may be negative and therefore the correct condition for the component of the force of friction is

$$-\mu R \leq F \leq \mu R.$$

For $F = \mu R$, the friction is said to be limiting and a body is said to be in limiting equilibrium and is on the point of slipping.

Once a body has started slipping, experiment shows that the friction force is still directly proportional to the normal reaction but the coefficient of proportionality is not always equal to μ and is sometimes denoted by μ' and is referred to as the coefficient of sliding friction. It is often found that $\mu' < \mu$, this explains the phenomenon of the book being harder to start sliding than to keep sliding and of a drawer which suddenly comes out with a rush.

In many cases the modelling assumption is that both the coefficients of friction are the same and, if you are not told specifically which coefficient of friction is given, then you should assume that you can use the same value for both limiting equilibrium and sliding. For steel on steel $\mu = 0.6$, $\mu' = 0.4$, for tyres on a dry road $\mu = 0.9$, $\mu' = 0.8$.

Miscellaneous Exercises 2

1 The components parallel to the x- direction of three forces acting at a point are 5 N, 7 N and 8 N. The components parallel to the y-direction of the forces are 4 N, 11 N and 6 N. Find

(a) the components in the x- and y- directions of the resultant of these forces,

(b) the magnitude of the resultant,

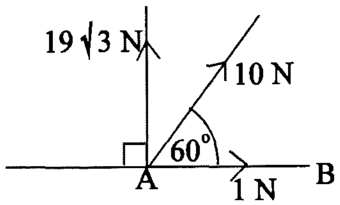
(c) the angle that the resultant makes with the x-direction,

(d) the magnitude and direction of the single additional force which will be in equilibrium with the other three forces.

2 The components parallel to the x-direction of three forces acting at a point are 1N,–5 N and p N. The components parallel to the y-direction of two of the forces are 1N, 3 N and the third has no component in the y-direction.

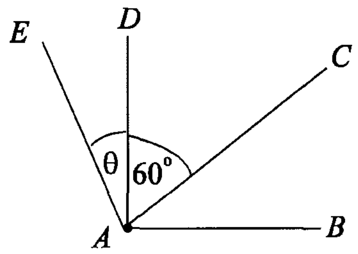
Given that the resultant of the forces has magnitude 5 N, find the two possible values of p.

3



Three forces of magnitudes $19\sqrt{3}$ N, 10 N and 1 N act at the point A in the directions shown in the above diagram. Find the magnitude of the single additional force acting at A which will produce equilibrium and find the angle between this force and the line AB.

4



A particle of weight 18 N is hanging in equilibrium at a point A supported by four strings AB, AC, AD, AE, all in the same vertical plane. The string AB is horizontal and the tension in AB is $7\sqrt{3}$ N; the string AC makes an angle of 60° with the vertical and the tension in AC is 8 N; the string AD is vertical and the tension in AD is 3 N; the string AE makes an angle of $\theta°$ with the vertical and the tension in AE is T. Find the values of T and θ.

5 A particle P of mass 0.2 kg is suspended in equilibrium from a fixed point O by a <u>light</u> <u>extensible</u> <u>string</u> <u>of</u> <u>natural</u> <u>length</u> 0.4 m. State which one of the words underlined enables you to assume that the tension is the same at all points of the string.

Given that the modulus of elasticity of the string is 10 N, find the distance OP.

6 A particle of weight 60 N is attached to two inextensible strings each of length 13 cm. The other ends of the strings are attached to two points A and B on the same horizontal level at a distance of 24 cm apart. Find the tension in the strings when the particle hangs in equilibrium.

The inextensible strings are then replaced by elastic ones, each of natural length 13 cm and of the same modulus of elasticity. The particle then hangs in equilibrium 9 cm below the line AB. Find the modulus of elasticity of the strings.

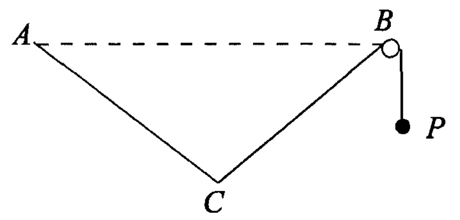
7 Forces of magnitude P and Q act along lines OA and OB respectively, and their resultant is a force of magnitude P; if the magnitude of the force along OA is changed to $2P$ the resultant is again a force of magnitude P. Find

(i) Q in terms of P,

(ii) the angle between OA and OB,

(iii) the angles which the two resultants make with OA.

8



The diagram shows a light inextensible string fixed at a point A and passing over a small smooth peg B fixed at the same level as A. A particle P, of mass m, hangs freely from the other end of the string. A smooth ring C, also of mass m, is free to slide on the string between A and B and the system is in equilibrium. Show that AC and BC are both inclined at an angle $\pi/3$ to the vertical.

9 A book of mass 1.5 kg rests on a rough plane inclined at an angle α to the horizontal. Given that the coefficient of friction between the plane and the book is 0.3 and that the book is on the point of slipping down the plane, find α.

10 A car is parked on a hill with its brakes locked. The car is of mass 1200 kg and the hill is inclined at an angle of 20° to the horizontal. Find the total friction force and normal reaction of the road on the car.

What modelling assumption do you make in your calculations.

11 The above car is then parked on a hill whose surface is such that the coefficient of friction between the tyres and the road is 0.6. Find the maximum slope such that the car will not slip down it.

12 A block of mass 3 kg rests on a rough horizontal table. When a force of magnitude 10 N acts on the block at an angle of 60° to the horizontal in an upward direction, the block is on the point of slipping. Find the coefficient of friction between the block and the table.

13 A book is placed on a desk lid which is slowly tilted. Given that the book starts to slide when the lid is inclined at an angle of 30° to the horizontal, find the coefficient of friction.

14 A particle is placed on a smooth plane inclined at an angle of 35° to the horizontal. The particle is kept in equilibrium by a horizontal force of magnitude 8 N acting in the vertical plane containing the line of greatest slope of the plane which passes through the particle. Find

(a) the weight of the particle,

(b) the magnitude of the force exerted by the plane on the particle.

15 A particle is suspended in equilibrium by two light inextensible strings and hangs in equilibrium. One string is inclined at an angle of 30° to the horizontal and the tension in the string is 40 N. The second string is inclined at an angle of 60° to the horizontal. Calculate in newtons

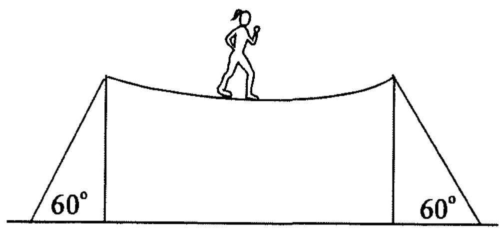
(a) the weight of the particle,

(b) the magnitude of the tension in the second string.

16 A particle P is placed on the inner surface of a fixed hollow sphere of centre O. Given that the coefficient of friction is 0.5 and that the particle rests in limiting equilibrium, find the tangent of the angle between the downward vertical and OP.

17 A particle of mass m is in equilibrium on a rough plane inclined at an angle α to the horizontal. When a force of magnitude mg, in the sense up the line of greatest slope, is applied to the particle, the latter is about to move up the plane. When a force of magnitude $\frac{mg}{2}$ in the sense down a line of greatest slope is applied to the particle, it is just about to move down the plane. Find $\sin\alpha$ and the coefficient of friction.

18



The diagram shows a circus artiste walking across a tightrope. The rope is tied at each end to a vertical pole and these poles are held in position by wires attached to their ends. The other ends of the wires are fixed to the ground and both wires are inclined at an angle of 60° to the horizontal. When the artiste, whose mass is 70 kg, is midway across the tightrope the tightrope is inclined at an angle of 80° to both poles. Find

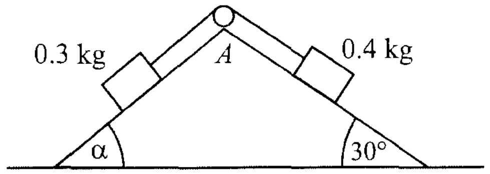
(i) the tension in the tightrope,

(ii) the tensions in the supporting wires,

(iii) the thrust exerted at the top end of each pole.

19 Two particles of the same mass are connected by an inextensible string. One particle lies on a rough plane inclined at an angle θ to the horizontal and the other hangs freely. The string connecting them passes over a smooth pulley which is above the particles and which separates the string into a part parallel to the inclined plane, and a vertical part. Show that the system will move when released from rest if the coefficient of friction between the plane and the particle is less than $\sec\theta - \tan\theta$.

20

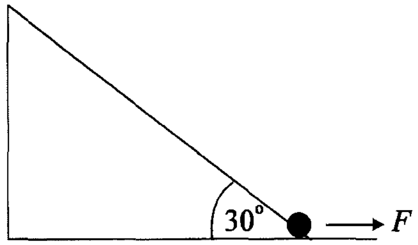


The diagram shows two particles, of mass 0.3 kg and 0.4 kg, in equilibrium on two smooth inclined planes intersecting at a point A. They are joined by a light string passing over a small smooth pulley at A and the particles and A are in the same vertical plane. Find

(i) the tension in the string,

(ii) the angle α.

21



The diagram shows a heavy particle of mass 0.3 kg in equilibrium on a smooth horizontal plane. The particle is attached to a fixed point by a light string inclined at an angle of 30° to the horizontal. A horizontal force of magnitude F is applied as shown.

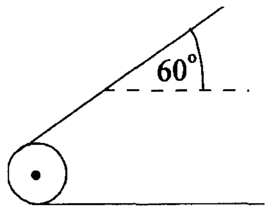
Given that $F = 4$ N find

(i) the tension in the string,

(ii) the normal reaction of the plane on the sphere.

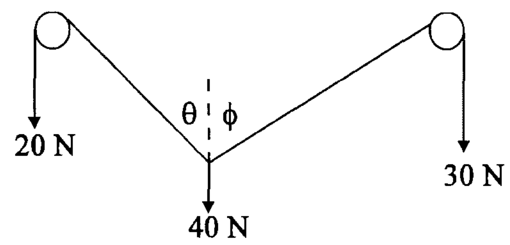
Discuss the behaviour of the reaction as F is increased.

22



The diagram shows a rope passing over a fixed pulley; the directions at which the rope leaves the pulley are as shown. Assuming that the tensions in both parts of the rope are the same, and that the total force on the pulley is 3 kN, find the tension in the rope.

23



The diagram shows two strings attached to a particle of weight 40 N, the strings pass over two smooth pulleys at the same horizontal level and carry particles of weight 20 N and 30 N, respectively, at their ends. Given that the particles are in equilibrium verify that $\theta = 46.6°$, $\phi = 29°$.

This kind of apparatus forms the basis for verifying the " triangle law".

It is possible to calculate the values of the angles corresponding to any set of weights but this requires more trigonometry than you yet know. You can get over this however, either by using scale drawing or using weights so that the strings at the point of intersection are perpendicular. (This will be true if the square of the weight at the point of intersection is the sum of the squares of the other weights.) You can then find the angles by resolving along the strings. If you can set up experiments with these weights you can compare your calculated values for the angles with the measured ones.

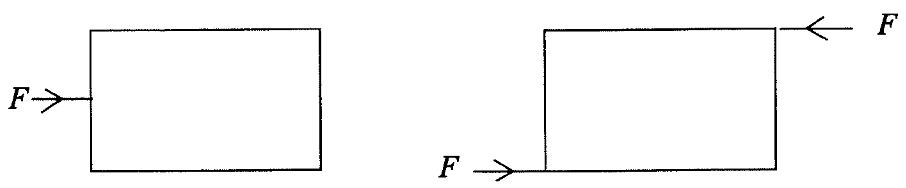
Chapter 3

Parallel forces acting on bodies

After working through this chapter you should be able to

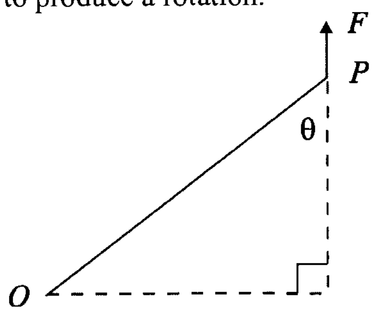
- find the moment of a force about a point,
- solve problems of the equilibrium of bodies acted on by parallel forces.

3.1 Moment of a force



(a) (b)

If you push a book along a table by applying a force perpendicular to, and at the middle point of, one edge, as shown in diagram (a) above, the book will move roughly in a straight line. On the other hand if you push the book by applying forces of the same magnitude, but of opposite directions, along two parallel edges, as in diagram (b) above, then the book will tend to rotate about its centre. If the book were pushed by applying a force at an arbitrary point of an edge then the motion would be a mixture of rotation and translation. Therefore, as the point of application moves, the effect of the force varies and in some circumstances there will be a turning effect. The moment of a force measures the tendency of a force to produce a rotation.



Definition

The magnitude of the moment, about a point O, of a force F acting through the point P is the product of the magnitude of the force and the perpendicular distance from O to the line of action of the force (this is the line through P in the direction of the force).

If the force is measured in newtons and the distance in metres, then the unit of the moment is the newton metre, abbreviated to Nm.

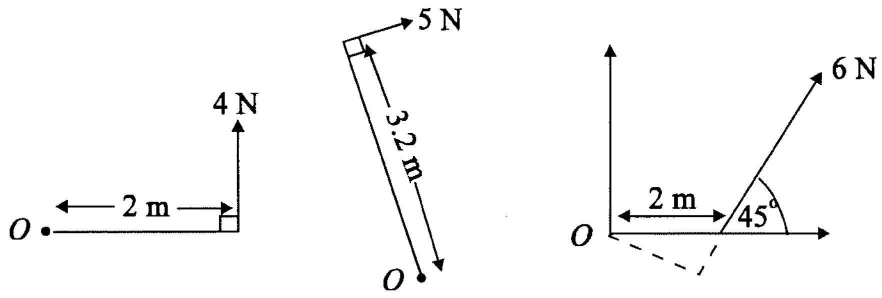
The moment about O is said to be "clockwise" or "anticlockwise" depending on whether the force is in the sense which would produce a clockwise or anticlockwise rotation about O. Moments of different forces often have to be added together and the convention used is that anticlockwise moments are positive and clockwise ones negative.

In the above diagram the perpendicular distance from O to the line of action is $OP \sin \theta$ so the moment is of magnitude $F \times OP \sin \theta$. For the sense shown in the diagram the moment is positive.

If the line of action of the force passes through O, then the moment will be zero.

Example 3.1

Find the moments of the following forces about O.



(a) (b) (c)

(a) The perpendicular distance of O from the line of action is 2 m, the sense of rotation is anti clockwise so the moment is 8 Nm.

(b) The perpendicular distance of O from the line of action is 3.2 m, the sense of rotation is clockwise so the moment is -16 Nm.

(c) The perpendicular distance of O from the line of action is $2 \cos 45^\circ$ m $= \sqrt{2}$ m, the sense of rotation is anti-clockwise so the moment is $6\sqrt{2}$ Nm.

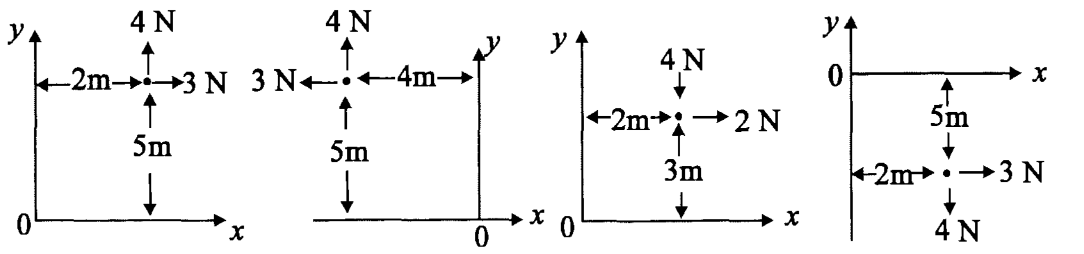
It is possible to prove that the moment of a force about a point is the sum of the moments, about that point, of the components of the force in any pair of directions. This often gives a much easier way of working out the moment of a force than trying to find the perpendicular distance from the line of action.

Example 3.2

Find the moment about O of the force with x- and y- components X N, Y N and acting at the point (a m, b m) when

(a) $X = 3$, $Y = 4$; $a = 2$, $b = 5$ (c) $X = 2$, $Y = -4$; $a = 2$, $b = 3$

(b) $X = -3$, $Y = 4$, $a = -4$, $b = 5$ (d) $X = 3$, $Y = -4$; $a = 2$, $b = -5$.



(a) (b) (c) (d)

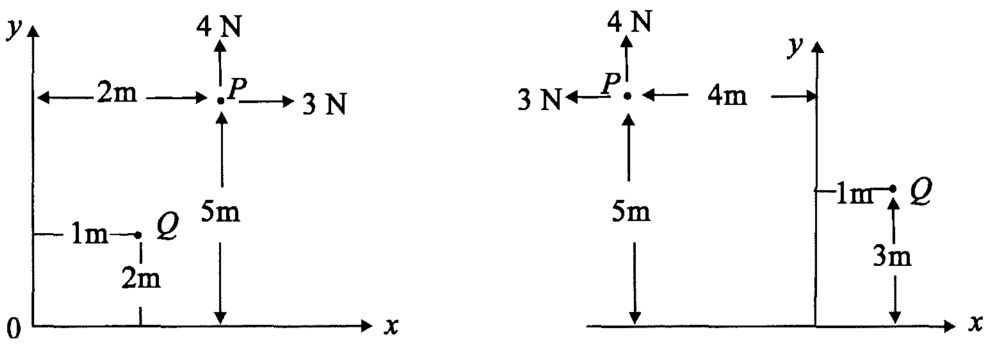
(a) The y-component has an anti clockwise moment of 4×2 Nm and the x-component has a clockwise moment of 3×5 Nm. The total moment is therefore
(8–15) Nm = –7 Nm.
(b) The y-component has a clockwise moment of 4×4 Nm and the x-component has an anti-clockwise moment of 3×5 Nm. The total moment is therefore
(–16+15) Nm =–1 Nm.
(c) The y-component has a clockwise moment of 4× 2 Nm and the x-component has a clockwise moment of 2×3 Nm. The total moment is therefore (–8 – 6) Nm = –14Nm.
(d) The y-component has a clockwise moment of 4?× 2 Nm and the x-component has an anti clockwise moment of 3×5 Nm. The total moment is therefore
(–8 +15) Nm = 7 Nm.

Example 3.3

Find the moment about the point Q (x_0 m, y_0 m) of the force with x and y components X N and Y N and acting at the point P (a m, b m) when

(a) $X = 3$, $Y = 4$; $a = 2$, $b = 5$, $x_0 = 1$, $y_0 = 2$

(b) $X = -3$, $Y = 4$; $a = -4$, $b = 5$, $x_0 = 1$, $y_0 = 3$.

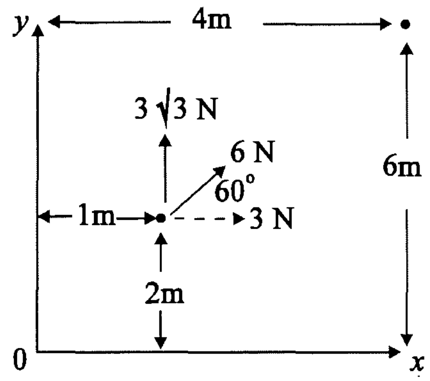


(a) (b)

(a) The y-component has an anti-clockwise moment of 4× 1 Nm and the x-component has a clockwise moment of 3×3 Nm. The total moment is therefore
(4 – 9) Nm = –5Nm.
(b) The y-component has a clockwise moment of 4× 5Nm and the x-component has an anti-clockwise moment of 3×2 Nm. The total moment is therefore
(–20 + 6) Nm = –14Nm.

Example 3.4

The line of action of a force of magnitude 6 N passes through the point (1,2) and the direction of the force makes an angle of 60° to the x-direction. Find the moment of the force about the point (4,6).



The force has components 3 N and $3\sqrt{3}$ N along the x- and y-axes, as shown in the diagram. The y-component has a clockwise moment of $3\sqrt{3} \times 3$ Nm whilst the x- component has an anti-clockwise moment of magnitude 3×4 Nm. The total moment is therefore $(12 - 9\sqrt{3})$ Nm = –3.59 Nm.
It is possible to derive a formula for the moment about the point (x_0, y_0) of the force with components (X, Y) acting at the point (x, y). The moment is $(x - x_0)Y - (y - y_0)X$. It is

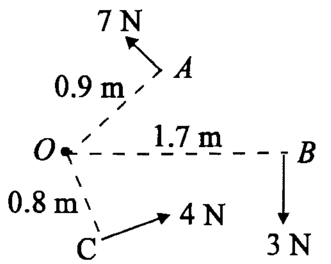
not worth trying to remember this (though you should try and derive it) but it can be useful in checking.

Moment of several forces

The moment about a point of several forces (not necessarily acting through one point) is defined to be the algebraic sum of the moments about the point of the individual forces.

<u>**Example 3.5**</u>

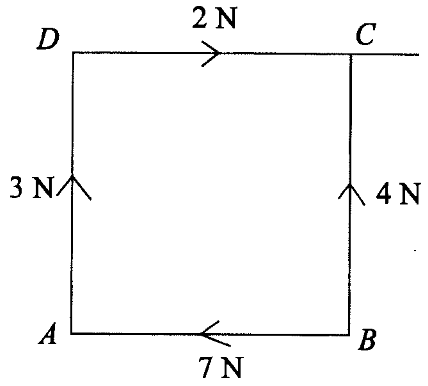
Find the moment about O of the following system of forces. The forces are all perpendicular to the dotted lines.



The force acting at A has an anti-clockwise moment of 6.3 Nm, that at B has a clockwise moment of 5.1 Nm and the force at C has an anti-clockwise moment of 3.2 Nm.
The total moment is therefore 6.3 + 3.2 – 5.1 Nm = 4.4 Nm.

<u>**Example 3.6**</u>

Forces of magnitude 7 N, 4 N, 2 N, and 3 N act in the senses shown in the diagram, along the sides BA, BC, DC and AD respectively, of the square $ABCD$ of side 3 m. Find the moment about A and about C of the system of forces.

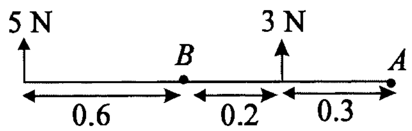


The forces whose lines of action pass through A will have no moment about A. The force along BC will have a anticlockwise moment of magnitude 4×3 Nm = 12 Nm, and the force along CD will have a clockwise moment of 2×3 Nm = 6 Nm. The total moment is therefore (12 – 6) Nm = 6 Nm.
The forces whose lines of action pass through C will have no moment about C. The force along BA will have a clockwise moment of magnitude 7×3 Nm = 21 Nm, and the force along AD will have a clockwise moment of 3×3 Nm = 9 Nm.
The total moment is therefore (–21–9) Nm = –30 Nm.

<u>**Exercises 3.1**</u>

In questions 1 to 4 forces are shown acting at various points on a straight line and the distances are measured in metres. Unless otherwise indicated, the forces are perpendicular to the line. Find the moments of the systems about the points A and B.

1.



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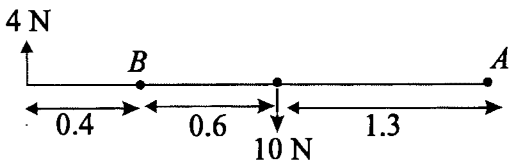

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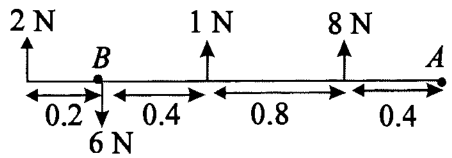

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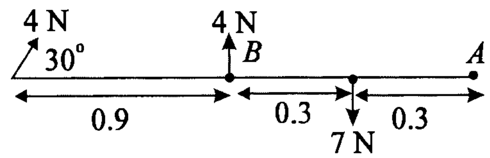
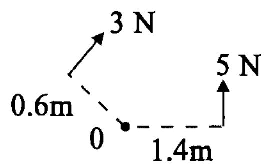


In questions 5 to 7 find the moments, about O, of the systems of forces shown. All forces are perpendicular to the dotted lines.

5.



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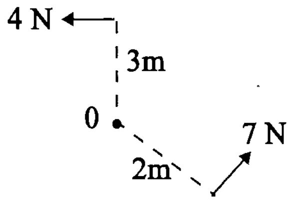

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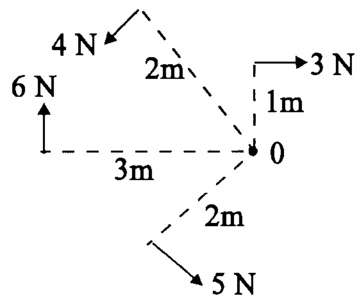


In the following three questions, the moments of the systems of forces given are to be found about the point with coordinates, in metres (a, b). The unit of force is the newton and the unit of distance is the metre.

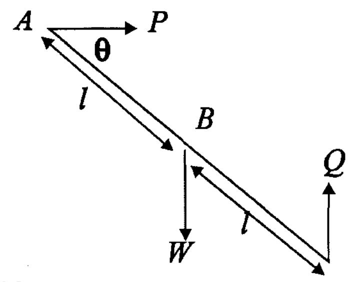
8 Force with components (2,3) acting at (1,1) and a force with components (5,4) acting at (3,–1), $a = 0$, $b = 0$.

9 Force with components (5,–3) acting at (–4,1) and a force with components (–2,–1) acting at (5,–1), $a = 2$, $b = 1$.

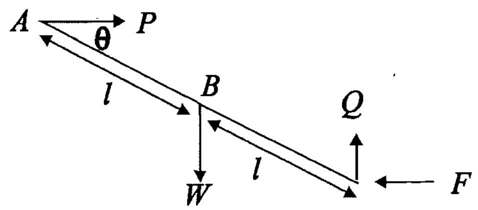
10 Force with components (3,1) acting at (2,–3) and a force with components (–5,2) acting at (–6,–1), $a = -3$, $b = 2$.

In the following questions, find the moments of the systems shown about the points A and B.

11.



12.



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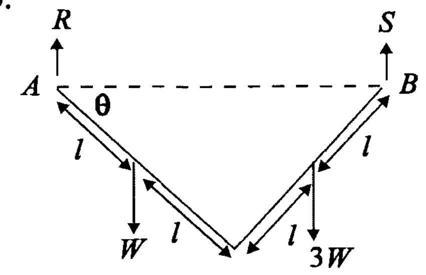
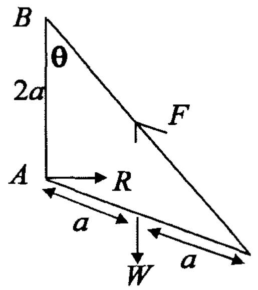


14.



3.2 Equilibrium of a body acted on by parallel forces

The conditions for the equilibrium of a body when a system of parallel forces acts are

(a) The sum of the components of all the forces is zero,

(b) The sum of the moments of all the forces about any point is zero (or, equivalently, the total clockwise moment is equal to the total anti-clockwise moment).

(Since the forces are all parallel, the components will all be parallel to one direction and therefore (a) effectively simplifies to requiring the component in one direction to be zero.)

You can replace both of these by

(c) The moment of all the forces about any two points is zero.

In most cases it is simpler to use (a) and (b) but you should always remember that you can only get two conditions per body.

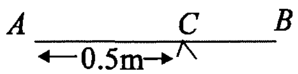
(If, for example, you got three equations by using (a) and (c) you would find that the third equation could have been obtained from the other two. The third equation therefore would not have given any more information).

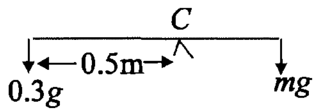
The important thing, again, is to draw clear force diagrams and then apply the conditions. If two bodies are in contact you may have to consider them separately but you must remember Newton's third law in making the force diagrams.

The choice of the point about which to take moments is again yours and you should try to choose it so that one unknown force is eliminated by taking moments about a point on the line of action of that force.

There are no new modelling assumptions made but you should remind yourself of the meanings of the phrases in the Glossary in Chapter 2. The only slightly different feature is that, for bodies that are not light, the force of gravity will act. This acts through the centre of gravity. You will not be expected, except in simple cases, to know the position of the centre of gravity. You should know that the centre of gravity of a uniform rod is at its midpoint.

<u>Example 3.7</u>





The diagram shows a light rod AB, of length 0.8 m, pivoted at a point C a distance 0.5 m from A. Find the mass that has to be placed at B, so that the rod will stay horizontal with a mass of 0.3 kg at A.

The force diagram is shown in the right hand diagram above.

Since there is a pivot at C it is better to take moments about it so that the reaction at the pivot will not enter into the moment equation. If the mass at B is denoted by m kg then equating the moments gives

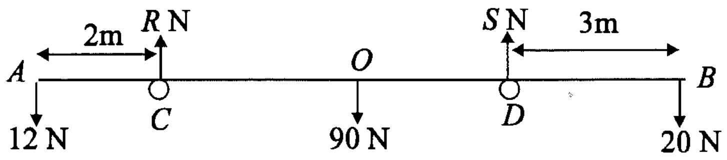
$$0.3 \times 0.5 = m \times 0.3$$

$$\text{so} \qquad m = 0.5.$$

Example 3.8

A light beam AB of length 8 m has a load weight of 90 N attached to its midpoint O. The beam rests horizontally on two smooth pegs C and D with $AC = 2$ m, $DB = 3$ m and with loads of weight 12 N and 20 N attached at the ends A and B respectively. Find the reactions at the pegs.

The force diagram is:



The pegs are smooth so their reactions are vertically upwards. Resolving vertically gives

$$R + S = 122.$$

If moments are taken about C, then the reaction R will not occur in the equation. Therefore

$$90\times 2 + 20\times 6 = 12\times 2 + S\times 3,$$

giving $S = 92$ and hence $R = 30$.

Example 3.9

Find, for the previous example, the greatest weight that can be placed at B without disturbing equilibrium.

Obviously for a sufficiently large weight at B the rod will start turning about D and when this happens $R = 0$. If the weight is denoted by W, then resolving vertically gives

$$R + S = 102 + W.$$

The moment equation now becomes

$$90 \times 2 + W \times 6 = 12 \times 2 + S \times 3.$$

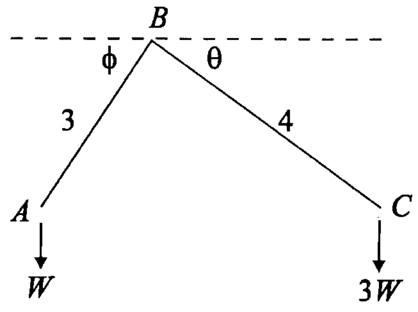
If $R = 0$ then, from the first equation, $S = 102 + W$, substituting this in the second equation gives $W = 50$.

In finding W a guess was made about what would happen. It is possible to avoid the guess with very little extra work. Solving the moment equation for S gives $S = 52 + 2W$, and this gives $R = 50 - W$. This shows that R would be negative for $W > 50$, so contact would have been lost.

Example 3.10

Two light rods AB and BC are rigidly connected at B so that they are at right angles to each other. AB is of length 3 m, BC is of length 4 m, and weights W and $3W$ are attached to A and C respectively. The configuration is suspended in equilibrium by a light string attached to B. Find the inclination of AB to the horizontal.

The force diagram is :



Taking moments about B gives

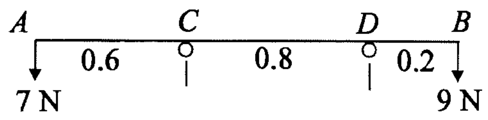
$$3W\cos\phi = 12\,W\cos\theta$$

Since $\theta + \phi = 90^\circ$, $\cos\phi = \sin\theta$ and therefore $\tan\theta = 4$ and $\theta = 76^\circ$.

Exercises 3.2

Questions 1 to 4 involve a light rod, simply supported at points C and D, and acted upon by the forces shown. The distances are measured in metres. Find the forces acting at C and D.

1.



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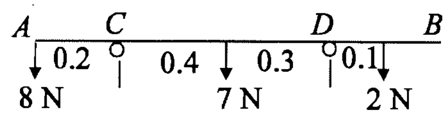

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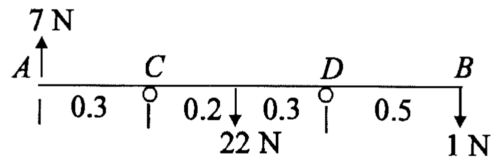

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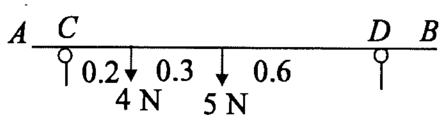


5 A uniform rod AB of length 4 m and mass 4 kg is pivoted about the point C where $AC = 1$ m. Find the mass of the particle which must be attached at A so that equilibrium is possible with the rod horizontal.

6 A uniform rod AB of length 6 m and mass 8 kg has a mass 12 kg attached at A and a mass 16 kg attached at B. Find the position of the point about which the rod can be balanced in a horizontal position.

7 A see-saw is made of a heavy plank of mass 30 kg and length 5 m and is pivoted at its midpoint. Two children of mass 25 kg and 35 kg can sit, one at each end of the seesaw, with the latter horizontal.

Find the distance of the centre of gravity of the plank from the child of mass 25 kg. State two modelling assumptions that you make.

8 A light rod AB of length 2 m is simply supported at points C and D, where $AC = 0.6$ m, $BD = 0.3$ m. A downward force of magnitude 60 N is applied at the midpoint of AB. Find the forces acting at the pegs. Find also the least force acting downwards at A which will disturb equilibrium.

9 A light square lamina $ABCD$ is free to turn about its centre in a vertical plane. A particle of mass m is attached at A. Find the mass of the particle that has to be attached at B so that the square can be in equilibrium, with AB inclined at an angle of 30° above the horizontal and A lower than B.

3.3 Resultant of a number of parallel forces

The idea of resultant can be extended to a number of parallel forces. If the sum of the components of all the forces is not zero, the resultant is a force whose component is the sum of the components of all the separate forces. It acts through a point such that the moment of the resultant about any point is the sum of the moments of all the other forces about that point.

The point through which the resultant acts is such that the sum of the clockwise moments about that point is equal to the sum of the anti-clockwise moments about that point.

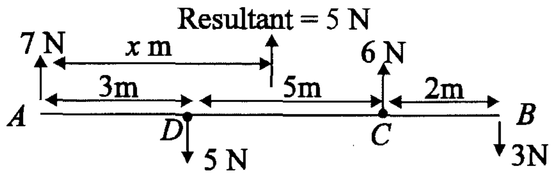
This definition is only valid when the sum of the components is not zero. If the sum of the components is zero, then the resultant is not a force but a couple. A couple is effectively two parallel forces, acting through different points, not on the same line, of equal magnitude but opposite directions.

You will not be expected to know anything about couples.

If you know the resultant of a system of parallel forces then you can obtain equilibrium by adding an equal and opposite force to the resultant.

Example 3.11

Find the resultant of the forces shown below.

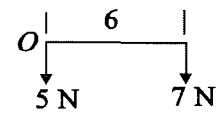


The component of the resultant up the page is $(7 + 6 - 5 - 3)$ N $= 5$ N. Taking moments about a point between D and C and at a distance x m from A, the anti-clockwise moments is $5(x - 3) + 6(8 - x)$, and the clockwise moment is $7x + 3(10 - x)$. Therefore $5(x-3) + 6(8 - x) = 7x + 3(10 - x)$ i.e $5x = 3$. The resultant is therefore a force of magnitude 5 N acting at a distance of 0.6 m from A.

Exercises 3.3

Find the resultant of the forces shown, the distances are all measured in metres.

1.



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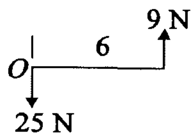

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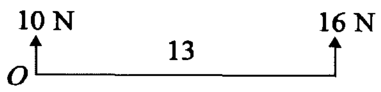

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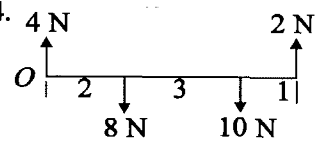

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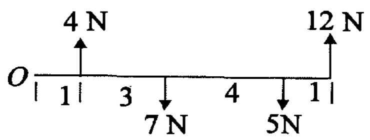
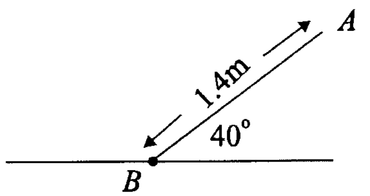


Miscellaneous Exercises 3

1



The diagram shows a light rigid rod AB of length 1.4 m inclined at an angle of 40° to the horizontal with the end B fixed to horizontal ground.

(a) Find the moment about B of a downward vertical force of magnitude 120 N applied at A.

(b) Find the magnitude and direction of the force of least magnitude that can be applied at A and whose moment about B is equal to that found in (a).

2 A uniform beam AB, of length 8 m and weight 200 N, rests horizontally on two smooth supports at points C and D, where $AC = 1$ m and $AD = 6$ m. Loads of 100 N and 400 N are attached to the beam at points E and F, where $AE = 2$ m and $AF = 5$ m. Find the reactions at the supports.

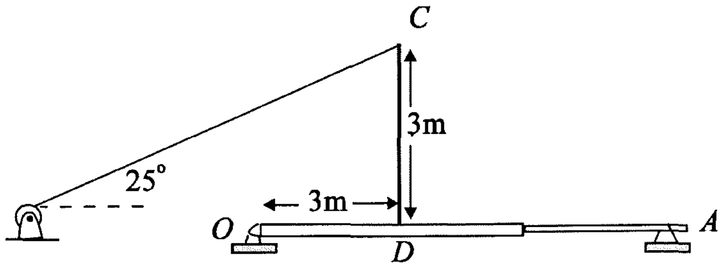
3 A uniform plank AB, of length 4 m and weight 200 N, rests horizontally on two smooth supports at points C and D, where $AC = 0.5$ m and $AD = 3.2$ m, with a load W N attached at B.

(a) Given that $W = 84$, find the reactions at the support.

(b) Find the greatest value of W for which equilibrium is possible.

4 A non-uniform beam AB, of length 10 m and weight 100 N, rests horizontally on two smooth supports at points C and D, where $AC = 3$ m and $AD = 8$ m. When a weight of 200 N is suspended from the midpoint of AB, the magnitude of the reaction at C is twice the magnitude of the reaction at D. Find the distance of the centre of gravity of the beam from A.

5



The diagram shows a flagpole OA with a rigid rod CD attached to it. A rope is attached to C and passes over a winch which is used to wind the rope and lift the flagpole. Given that the tension in the rope is 3 kN, find the moment of this tension about O.

6 A motorcycle has mass 200 kg, the points of contact of the wheels with the road are 1.5 m apart and the line of action of the weight is at the same distance from both wheels. The rider is of mass 75 kg and the line of action of his weight intersects the road at a distance of 0.9 m from the front wheel. Find the reaction of the road on the two wheels.

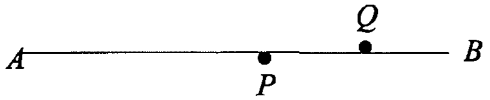
7 A woman of mass 75 kg crosses a garden stream by using, as a bridge, a plank of length 3 m and mass 150 kg. Assume that

- the plank can be modelled as a rod with its centre of gravity at its centre,
- the woman can be modelled as a particle,
- the reactions of the banks of the stream act at the ends of the plank.

Find the reactions of the banks on the plank given that the woman is 0.5 m along the plank.

What further information would you need to find the reactions when the woman is pushing a wheelbarrow along the plank?

8



The diagram shows a light rod AB of length $8a$ resting horizontally between two smooth pegs P and Q, where $AP = 5a$ and $QB = a$. A particle of weight $4W$ is attached at the midpoint of the rod. Find the reactions of the pegs.

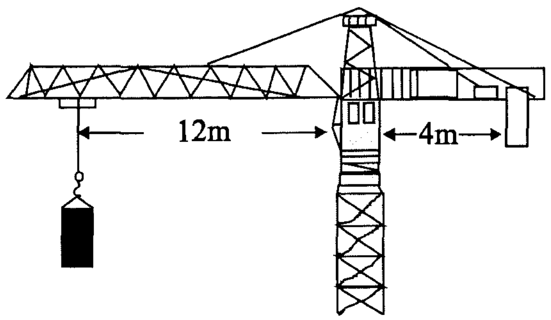
The greatest force that the peg Q can sustain is $10W$. Find the greatest magnitude of the force that can be applied at B, (a) downwards, (b) upwards, without breaking equilibrium.

9 The centre of gravity of a van is 2 m in front of its rear axle and 1 m behind its front axle. The van weighs 15 kN and carries a load of W kN whose centre of gravity is 0.5 m in front of the rear axle. The van is at rest on level ground.

(a) Find, assuming $W = 3$, the reactions on the axles of the van.

(b) Find W when the reactions on the two axles are equal.

10



The diagram shows a tower crane carrying a load of mass 25 tonne. Find, assuming that the mass of the cabin and the structure may be neglected, the mass of the counterweight necessary to achieve equilibrium. Find also, assuming the load to be increased to 80 tonne, the new position of the load.

11 A light rod $ABGCD$ with a load applied at G rests horizontally on two smooth supports at B and C. The lengths AB, BC and CD are 1.2 m, 2.4 m and 1.8 m respectively. The rod just starts to tilt when a load of 150 N is attached at A or when a load of 40 N is applied at D. Find

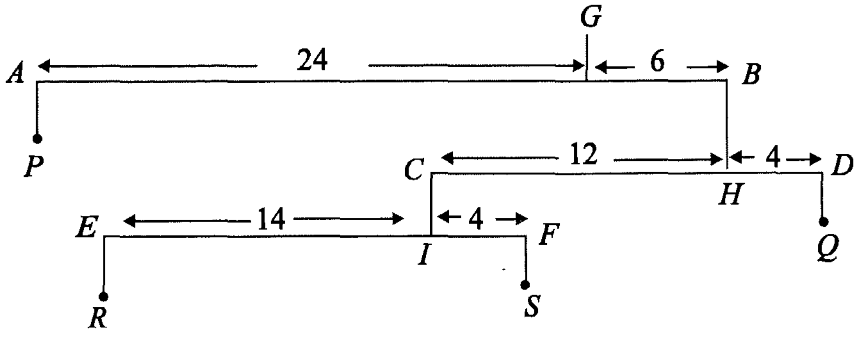
(a) the load at G,

(b) the length AG,

(c) the reactions on the rod when loads of 150 N and 60 N are simultaneously applied at A and D.

12 A light beam AB of length $6a$ rests horizontally on smooth pegs at its points of trisection. A heavy particle is placed at a point P of the beam. When a mass of 1 kg is placed at A, the beam just tilts. When a mass of 1 kg is placed at B the reactions at both supports are equal. Find the mass of the particle and the distance AP.

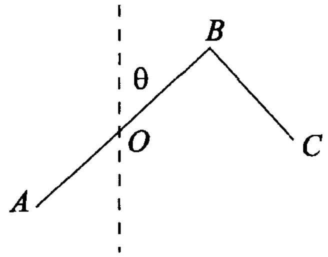
13



The diagram shows three light rods AB, CD and EF with particles P, Q, R, S suspended from A, D, E and F respectively. Light strings BH and CI, as shown, connect the three rods. The system is suspended in equilibrium from the point G on AB and, in

equilibrium, the rods are horizontal. Given that the weight of Q is 13.5 N, find the weights of the other particles. All the distances shown are in centimetres.

14



The diagram shows a light rod AB of length $4a$ rigidly joined at B to a light rod BC of length $2a$ so that the rods are perpendicular to each other and in the same vertical plane. The centre O of AB is fixed and the rods can rotate freely about O in a vertical plane. A particle of mass $4m$ is attached at A and a particle of mass m is attached at C. The system rests in equilibrium with AB inclined at an acute angle θ to the vertical as shown. By taking moments about O find the value of θ .

15 A light beam AD of length $6a$ rests horizontally on smooth pegs at its points of trisection B and C. A particle of weight W is placed at its midpoint. A man of weight $4W$, in order that he can stand at A, places a counter weight W' at a point P, between C and D, and at a distance x from D. Find the least value of W'.

Find also the maximum value of W' so that the beam will still be in equilibrium when the man is not standing on it.

16 A motor cycle which is too long to go on a scales is weighed by placing first one wheel and then the other wheel on the scales. The platform of the scales is above the level of the ground.

During the weighings the lower wheel is in contact with the ground and clamped so that the motor cycle remains vertical.

The distance between the centres A and B of the wheels (which are of equal diameter) is c. When the motorcycle is on level ground, the weight acts through a line which passes through the mid point of AB and the centre of gravity is at a height h above this midpoint. The inclination of AB to the horizontal when either wheel is on the scales is θ. Show that the sum of the two weighings is less than the actual weight W by $\frac{2Wh}{c}\tan\theta$.

Chapter 4

Kinematics of Rectilinear Motion

After working through this chapter you should

- understand what is meant, in rectilinear motion, by displacement, velocity and acceleration,
- be able to derive the "constant acceleration formulae" and use them to solve problems involving motion under constant acceleration,
- be able to solve simple problems of vertical motion under gravity.

4.1 Basic definitions

When trying to solve any problem involving the rectilinear motion (that is motion in a straight line) of a particle, it is essential at the outset to choose a particular direction to be the positive direction and to refer everything to this direction. The choice of reference direction does not matter; the important thing is to stick to the same reference direction throughout a given problem. Failing to do this is the greatest source of error in problems, particularly in setting up equations of motion. For simplicity in what follows, motion will be assumed to be along the x-axis and the positive direction to be that of increasing x.

Displacement

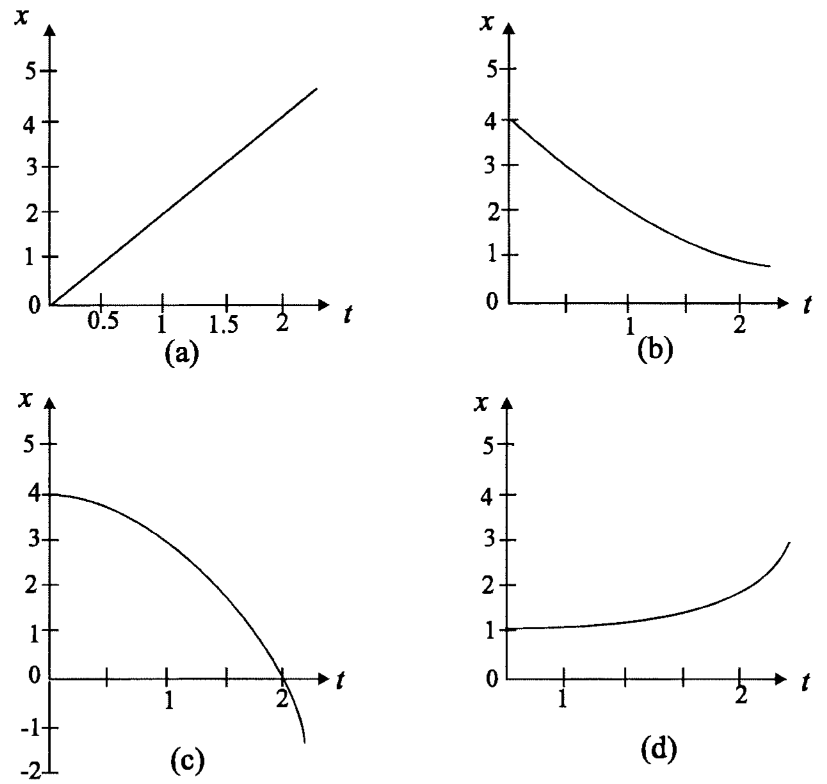
The position of a particle at any time is determined by its x-coordinate and in kinematics this coordinate is referred to as the displacement of the particle from the origin. Of course, the displacement x can be positive or negative depending on whether the particle lies to the right or the left of the origin. The distance of the particle from O is $|x|$. By definition, the displacement of a particle specifies its position uniquely, whereas the distance from the origin does not do this because it does not identify the side of the origin on which the particle lies.

The usual unit of measurement in the S I system is the metre, (abbreviation m) though for small distances the centimetre (cm) is used and, for large distances, the kilometre (km) is used.

In principle it is possible, by measuring the displacement of a particle, to express x as a function of t and diagrams (a), (b), (c) and (d) below show the behaviour of x as t varies for the four cases

(a) $x = 2t$, (b) $x = \frac{1}{4}(t-4)^2$, (c) $x = 4 - t^2$, (d) $x = \frac{1}{4}(t^3 - t^2) + 1$.

You will see that, in (c), x becomes negative for $t > 2$.



It is worth looking more closely at diagram (a) where $x = 2t$. For simplicity, it will be assumed that the displacement is measured in metres and time in seconds. The difference between the displacements at times T and $T + t'$, is $2t'$ for all values of T, i.e. the same distance is travelled in the same time interval. You can also see this from the diagram. Therefore in any interval of 1 second the particle travels 2 m and you would intuitively interpret this as saying that the particle has a speed of 2 ms^{-1}. It is not possible to give a simple interpretation in the other diagrams since you can see that equal distances are not covered in equal times. To cover these cases it is necessary to have a more sophisticated approach as follows.

Velocity

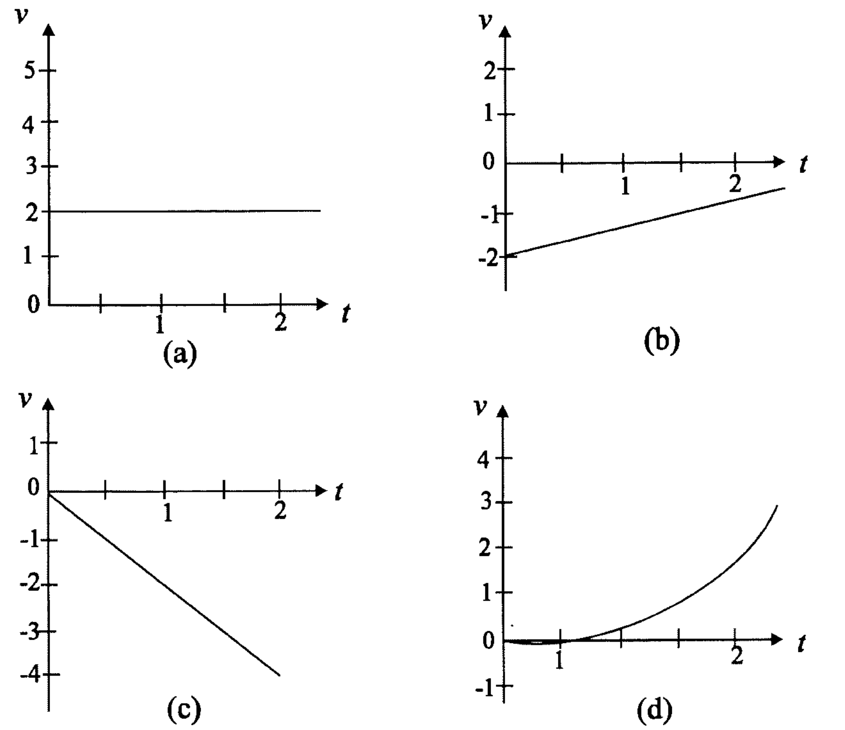
The velocity v in the positive direction is defined to be the rate of change of the displacement with respect to time, that is,

$$v = \frac{dx}{dt} = \dot{x}.$$

Superscript dots are often used to imply differentiations with respect to time t.

The unit of velocity commonly used in the S.I. system is the metre per second (ms^{-1}), and for velocities of relatively large magnitude the kilometre per hour (kmh^{-1}) is used.

If you have not yet come across differentiation in your course then there is a simple graphical way of defining the rate of change. This is that the rate of change of x with t at any time is the slope of the graph of x against t at that particular time. The diagrams below show the velocities corresponding to the four graphs of displacement above.



The graphs have all been drawn using differentiation and the functions defined are 2, $\frac{1}{2}(t-4)$, $-2t$ and $\frac{1}{4}(3t^2-2t)$. If you have not yet met differentiation you can check the graphs by drawing the tangents to the graphs of x as a function of t and see that the slopes are the values of v in the second set of graphs. In diagrams (b) and (c) the velocity is negative, showing, as you can check from the corresponding diagrams for x, that x decreases with time.

Kinematics of Rectilinear Motion

The velocity can again be either positive or negative and there is no direct dependence between the signs of x and v. For example, $x = 1-t^2$ is positive for $0<t<1$, whereas the velocity, which is $-2t$, is negative for $0<t<1$. Also in diagram (b) above, the velocity is negative though x is positive. All that is implied by a negative velocity is that the motion is in the opposite direction to the reference one. The speed is the magnitude of the velocity, that is, speed is equal to $|v|$. Velocity and speed as defined can vary with time but in diagram (a) above the velocity is constant, or uniform. In many practical situations the speed is estimated by dividing the total distance by the total time, this is however only a very rough estimate and should not be used to calculate the velocity at a particular time. The ratio distance/total time defines the average speed.

Acceleration

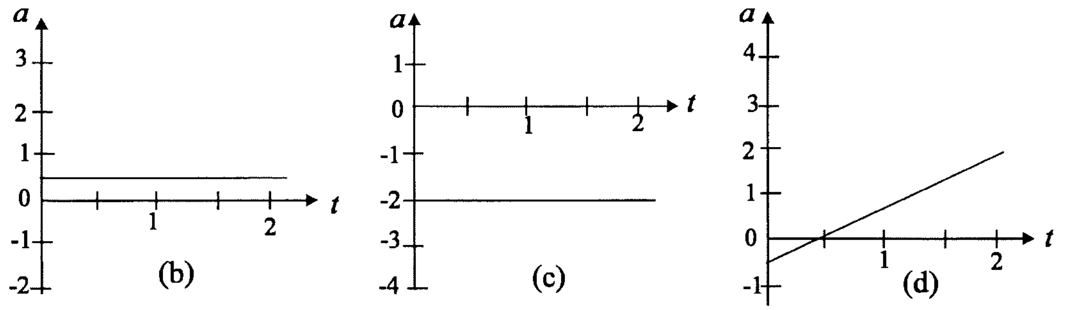
In diagrams (b) and (c) above, the slopes are constant, and therefore the rate of change of velocity can be worked out easily. This is not true for diagram (d) and in order to cover this, a further important idea has to be introduced, namely that of acceleration.

The acceleration a in the positive direction is defined as the rate of change with respect to time of the velocity in the positive direction, that is,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x}$$

Again this can be positive or negative. The units of acceleration are metres per second^2 (ms^{-2}) and kilometres per hour^2 (kmh^{-2}). When there is no possibility of misunderstanding, the phrase 'in the positive direction' is omitted after velocity and acceleration, but it should be remembered that it is always implied. If a particle is said to be moving with retardation r, then conventionally this means that the acceleration in the reference direction is equal to $-r$. A retardation is sometimes called a deceleration and a particle is sometimes said to decelerate.

The acceleration is the slope of the graph of v against t. For diagram (a) above the acceleration is zero and for diagrams (b), (c) and (d) the graphs of acceleration against time are given below.



The accelerations in the three cases are $\frac{1}{2}$, -2 and $\frac{1}{2}(3t - 1)$ and again you can check them by finding the slopes of the previous sets of curves. In the first two cases the acceleration is constant, the second case corresponding to a retardation.

In most practical situations the acceleration a will be given and may depend on all three of t, x and v and x or v have to be found. Finding x from a requires the solution of a differential equation; some simple examples of the general type of problem are discussed in 4.4 but it is possible to make considerable progress with the case of constant acceleration without elaborate calculation.

4.2 Constant acceleration

For a particle moving under constant (or uniform) acceleration a, with initial velocity (i.e. its velocity when $t = 0$) u, it is possible to give formulae for its displacement s from its initial position and its velocity v at time t. These are

$$v = u + at, \qquad \ldots\ldots (1)$$

$$s = ut + \frac{1}{2}at^2, \qquad \ldots\ldots (2)$$

$$v^2 = u^2 + 2as, \qquad \ldots\ldots (3)$$

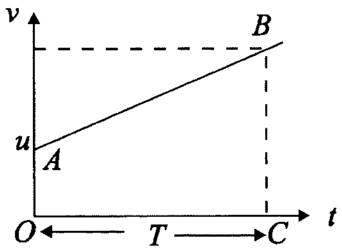
$$s = \frac{1}{2}(u + v)\,t. \qquad \ldots\ldots (4)$$

If the initial conditions were given at $t = T$ and not at $t = 0$ then t would have to be replaced by $t - T$. The derivation of these equations is given in 4.3.

Equations 1 to 4 are sufficient to enable all problems involving motion with constant acceleration to be solved, and they should be committed to memory.

It is also very important to remember that they can only be used for <u>constant acceleration</u>.

Before describing the use of these equations in solving problems, it is useful to give a graphical interpretation of equation 1 which is very useful in solving particular types of problems. The behaviour of v, as defined by equation 1, with t is shown below.



It is a straight line with gradient a. The area of the region between the t-axis and under the line, and between the lines $t = 0$ and $t = T$, is the area of the trapezium $OABC$ and this is $\frac{1}{2}(u + v)\,T$ which, by equation 4, is equal to s. This is a special case of the general result (shown in 4.3), that the area under the graph of v against t is equal to the distance covered.

Effectively, all the information supplied by equations 1, 2 and 4 is contained in the above diagram when the area $OABC$ is interpreted as s. Such a graph can be a useful, compact way of setting down the conditions in a given problem and is referred to as the velocity - time, or $v - t$, diagram.

Problem solving

The simplest kind of problem for motion under constant acceleration is the one in which u and a are given, and values of s and v are required for particular values of t. These problems are solved by substitution in equations 1 to 4. In solving problems it is important that all the quantities in equations 1 to 4 are evaluated in the same unit system. In the following examples, the S.I. unit system will be used so that the displacement from the initial position is denoted by s m, the initial and final speeds are denoted by u ms^{-1} and v ms^{-1} respectively, the acceleration by a ms^{-2} and the time by t s. This means that there are no units associated with s, u, v, a and t and they are, therefore, just numbers which satisfy equations 1 to 4.

<u>Example 4 .1</u>

Initially, a particle P moving with uniform acceleration 6 ms^{-2} has a velocity at the point O of 3 ms^{-1}. Find its velocity after 2 s and its displacement from O after 4 s .

This is the simplest kind of problem, since both u (= 3) and a (= 6) are given. The velocity after 2 s is found by setting $t = 2$ in equation 1, giving

$$v = 3 + 12 = 15,$$

so the velocity is 15 ms^{-1}. The distance travelled in the first 4 s is found by substituting $t = 4$ into equation 2, giving

$$s = 3 \text{ x } 4 + 3 \text{ x } 6 = 60.$$

The displacement from O is therefore 60 m.

A slightly harder class of problem arises when u and a are not given directly, but sufficient information is available to find them. In solving this kind of problem, the best method is to list the unknowns and then find them systematically by choosing whichever of equations 1 to 4 contains only one unknown. This equation then gives that unknown. First try to find u and t. Then s and v can be found for all values of t by using equations 1 to 4.

Example 4. 2

The velocity of a particle P moving with a uniform acceleration of 3 ms^{-2} increases from 2 ms^{-1} to 8 ms^{-1} as P moves from A to B. Find the distance between A and B.

In this case $v = 8$, $u = 2$, $a = 3$, s is required and this suggests using equation 3. Making the appropriate substitutions gives

$$64 = 4 + 6s,$$

so that $s = 10$. The points A and B are therefore 10 m apart.

This problem could also have been solved by substituting in equation 1 to find the total time, giving $t = 2$, and then using equation 4 to obtain the value of s.

Example 4.3

A train starts from rest at a station and moves with constant acceleration. Twenty seconds later it is moving with speed of 72 kmh^{-1} when it passes a signal box. Find the distance, in metres, between the station and the signal box.

In this question two units of length and two units of time are involved so the first thing is to decide on the units to be used and, since an answer is required in metres, it seems reasonable to use metres and seconds.

$$72 \text{ kmh}^{-1} = \frac{72 \times 1000}{3600} \text{ ms}^{-1} = 20 \text{ ms}^{-1}.$$

In this question t (=20), u (=0), v (=20) are given so a can be found from equation 1 and

$$20 = a20,$$

so that $a = 1$. The distance can now be found from equation 3, i.e.

$$400 = 2 \times 1 \times s,$$

so that the distance from the station to the signal is 200 m.

Example 4.4

The displacement from its original position of a particle moving with uniform acceleration is 6 m after 2 s and 20 m after 4 s. Find the displacement 6 s after the start of the motion.

Neither a nor u is given, but values of s are given for two values of t i.e. $s = 6$ when $t = 2$ and $s = 20$ when $t = 4$. Substituting into equation 2 for these two values gives

$$6 = 2u + 2a, \quad 20 = 4u + 8a.$$

Solving these simultaneously gives $a = 2$ and $u = 1$. The displacement after 6 s is found by substituting these values in equation 2, with $t = 6$, giving $s = 6 + 36 = 42$, and therefore the required displacement is 42 m.

Example 4.5

A boy moving up a hill on a skate-board experiences a retardation of magnitude 2 ms^{-2}. His speed at the bottom of the hill was 8 ms^{-1}, find how far up the hill he travels before coming to rest.

Since the boy experiences a retardation his acceleration in the sense up the hill is -2 ms^{-2}. In this case $u = 8$, $v = 0$ and $a = -2$ so s can be found from equation 3 i.e.

$$0 = 64 - 2 \times 2s,$$

giving $s = 16$ so that the boy travels 16 m up the hill before coming to rest.

Example 4.6

A car, travelling at 20 ms^{-1}, has to be braked suddenly and skids a distance of 25 m before stopping. Find the acceleration, assuming that it is constant, and the time taken to stop.

In this case $v = 0$, $u = 20$ and $s = 25$ so equation 3 can be used to find a i.e.

$$0 = 400 + 2 \times 25\, a,$$

giving $a = -8$, so that the acceleration is -8 ms^{-2}. This is a retardation and is to be expected since the car is slowing down. The time can now be found from equation 1 which gives

$$0 = 20 - 8t,$$

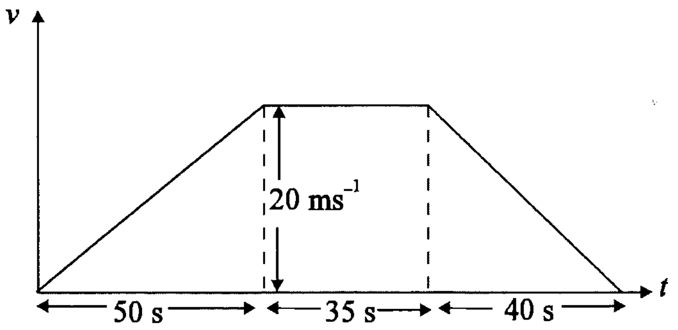
giving the time to be 2.5 s.

Probably the most complicated problems involving constant acceleration are those in which the acceleration is constant for a particular period but then switches to another constant value for a different period. This kind of problem can occur, for example, in the motion of a train which accelerates from rest to a steady speed, keeps that steady speed for a while, and then retards to come to rest. In such problems, equations 1 to 4 have to be applied systematically for each period, and the information given in a question used to find all the unknowns. It is in these problems, where the given information can be complicated, that the v - t diagram discussed is most useful. The given information can be displayed compactly on a diagram and elementary geometry used to complete the question. This approach is particularly useful for 'rest-to-rest' problems. The graph of v

against t will be a series of straight lines, with the parts with constant velocity being segments parallel to the t-axis.

Example 4.7

Starting from rest, a train moves with uniform acceleration and reaches a maximum speed of 20 ms^{-1} in 50 s. It runs at this speed for 35 s and then comes to rest with uniform retardation in 40 s. Find the total distance travelled.

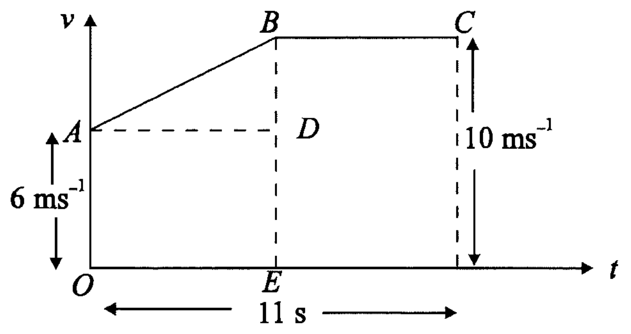


The diagram shows the information set out on a v - t diagram. The distance is the total area of the figure. The parallel sides of the trapezium are of lengths 125 and 35 respectively. The area is therefore $\frac{1}{2}(125 + 35) \times 20 = 1600$ so that the distance travelled is 1600 m.

Example 4.8

Over a 100 m length, a runner accelerates uniformly from 6 ms^{-1} to 10ms^{-1} and then maintains the latter speed over the remaining length. Given that the total time for the 100 m distance is 11 s, find the acceleration.

The v - t diagram is shown below.



The total area under the lines AB and BC is 100, this area is also equal to

$$\frac{1}{2}(6 + 10)\, OE + 10(11 - OE)$$

and equating this expression to 100 gives $OE = 5$.

The gradient of AB is the unknown acceleration a ms^{-2}, so that

$$a = \frac{BD}{AD} = \frac{10-6}{5} = 0.8.$$

Therefore $a = 0.8$ so that the acceleration is 0.8 ms^{-2}.

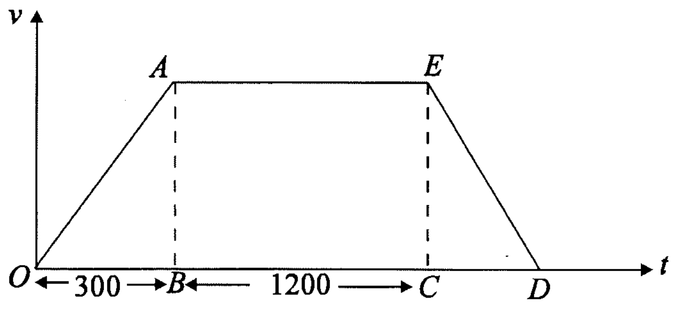
Example 4.9

A train starting from rest moves with constant acceleration for 5 minutes. It then moves at a constant speed for 20 minutes. A constant retardation is then applied, whose magnitude is twice that of the acceleration, until the train comes to rest. Find, given that the train travels 4.5 km whilst accelerating,

(i) the acceleration,

(ii) the total distance travelled.

The v-t diagram is given below, where time is measured in seconds and velocity in ms^{-1}.



(i) The distance travelled whilst accelerating is the area under the line OA. The length OB is 300 (corresponding to the time accelerating) so

$$4500 = \frac{1}{2} \times 300 \times AB,$$

giving $AB = 30$, i.e the constant speed is 30 ms^{-1}.

The slope of OA is $\frac{AB}{OB} = \frac{30}{300} = \frac{1}{10}$, so that the acceleration is $\frac{1}{10}\text{ms}^{-2}$.

(ii) The retardation is the slope of $DE = \frac{EC}{CD} = \frac{30}{CD}$, this is given to be twice the retardation so it is $\frac{1}{5}$ ms^{-1} so $CD = 150$ and the distance travelled whilst retarding, which is the area under DE, is $\frac{1}{2} \times 30 \times 150$ m i.e. 2.25 km.

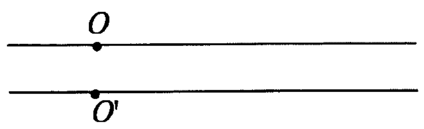
The length BC is 1200, corresponding to the time the train is travelling at constant speed, and therefore the distance travelled at constant speed is 30×1200 m = 36 km. The total distance travelled is therefore 4.5 km + 2.25 km + 36 km = 42.75 km.

Exercises 4.1

Questions 1 to 11 refer to a particle moving on a straight line with constant acceleration a ms^{-2}, so that at time t s the velocity and the displacement of the particle from a fixed point O are given by v ms^{-1} and s m, the initial velocity of the particle being u ms^{-1}. The acceleration, velocities and displacement are with respect to the same reference direction.

1 $a = 4, v = 12, u = 4$, find t.
2 $a = -5,\ u = 3, v = -12$; find t and s.
3 $u = 6, t = 2, s = 20$; find v.
4 $u = -2, a = 3, t = 5$; find s.
5 $a = 4, t = 4, s = 42$; find u.
6 $v = 11, u = 7, t = 2$, find a.
7 $v = 8, u = 6, s = 7$; find a.
8 $s = 25, u = 4, v = 11$; find t.
9 $s = 70, u = 4$, t = 5; find a.
10 $s = 8, u = 9, a = -5$; find the possible values of t.
11 $s = 25$ when $v = 13$ and $s = 52$ when $t = 4$; find the possible values of u and a.

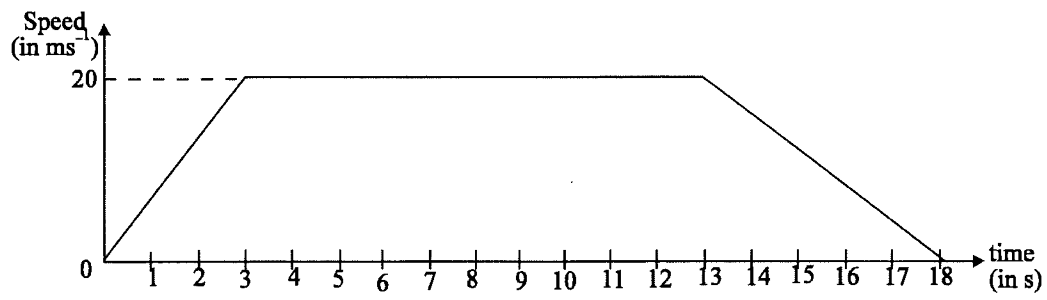
Questions 12 to 13 refer to particles P and Q free to move on adjacent parallel lines as shown in the diagram, O and O' are fixed points on the lines and OO' is perpendicular to both lines.



12 At time $t = 0$ s, P passes through O moving towards a point B with a speed of 7 ms^{-1} and thereafter moves with a constant acceleration of 2 ms^{-2} directed towards B. At the same time, Q passes through the point B', which is directly opposite to B and which is 114 m away from O', with a velocity directed towards O' of 3 ms^{-1} and thereafter Q has acceleration 1 ms^{-2} in the direction $B'O'$. Find when P and Q are level with each other.

13 At time $t = 0$ s, P passes through O with a speed of 4 ms^{-1} and thereafter moves with a constant acceleration of 2 ms^{-2}. At time $t = 2$ s, Q passes through the point C', 100 m away from O', with a velocity directed towards O' of 17 ms^{-1}. Thereafter Q has an acceleration of 4 ms^{-2} in the sense $C'O'$. (The distance between the tracks may be neglected). Find when P and Q are at a distance of 60 m apart.

14



The diagram is a v-t diagram for a particle which moves with constant acceleration for 3 seconds, then moves with a constant velocity for 10 seconds and then moves with constant retardation until it comes to rest. Find

(i) the acceleration,
(ii) the retardation,
(iii) the total distance travelled.

15 An underground train covers 576 m from rest to rest in 60 s. At first it has a constant acceleration of 0.5 ms^{-2}, then moves with constant speed and finally has a constant retardation of 1 ms^{-2}. Find the time taken for each stage of the journey.

16 A train approaching a station travels two successive distances of 0.25 km in 10 s and 20 s respectively. Assuming the retardation to be uniform (i.e constant), find the further time taken before coming to rest.

17 A train starting from rest is uniformly accelerated during the first 80 s of its journey in which it covers 600 m. It then runs at a constant speed until it is brought to rest in a distance of 750 m by applying a constant retardation. Find the maximum speed of the train and the magnitude of the retardation.

18 A car starting from rest moves with constant acceleration of 1 ms^{-2} for 10 seconds, it then moves at a constant speed for 1 km and then a constant retardation is applied to bring it to rest. The total distance travelled is 2 km. Find the maximum speed of the car and the total time taken.

19 A car moves off from rest with a constant acceleration of 1.2 ms^{-2} but after 12 s the driver sees an obstacle and is forced to apply the brakes which produce a retardation of 1.6 ms^{-2}. Find the total time taken from rest to rest.

4.3 Vertical motion under gravity

It is an observed fact that particles free to move in a vertical direction near the earth have the same constant acceleration, denoted by g (9.8 ms^{-2}) downwards. Therefore, all problems involving such motion can be solved by the methods described above. If s is measured upwards from the point of projection then the previous formulae hold with $a = -g$, whilst if s is measured downwards, a has to be replaced by g. It does not matter whether the upwards or downwards direction is taken as positive. Normally it is more sensible to measure s upwards for particles projected upwards, and downwards for those released from rest. Anything projected up will, of course, come down and in such cases, taking the positive direction of s to be upwards for the complete motion avoids sign errors.

A particle projected up with speed u will, by equation 3, have a speed of $v = \sqrt{u^2 - 2gs}$ when at a height s. Therefore, the maximum height h reached will be when the speed, v, is zero, that is when

$$u^2 = 2gh \text{ or } h = \frac{u^2}{2g} \qquad \ldots\ldots (5)$$

Therefore, a particle projected upwards with a speed u will rise a distance $u^2/2g$. Its speed on the downward path at the point of projection i.e. $s = 0$ is $v = \sqrt{u^2}$ and is again u. So, the velocity when it returns to the point of projection is equal in magnitude but opposite in direction to the original velocity of projection.

<u>Example 4.10</u>

A stone is thrown vertically up with speed 20 ms^{-1} from the top of a building 22.5 m high. After what time and with what speed will it strike the ground?

The upward displacement s m at time t s is, using equation 2,

$$s = 20\,t - \frac{1}{2}9.8\,t^2.$$

It is required to find the time when the stone is a distance 22.5 m below its original position, that is, when s = −22.5. Therefore,

$$-22.5 = 20t - 4.9\,t^2, \quad \text{that is,} \quad 4.9\,t^2 - 20t - 22.5 = 0.$$

The left hand side factorises as $(t - 5)(4.9t + 4.5)$, the positive root of the equation is 5, so the time taken is 5 s. The speed v when $s = -22.5$ can be found by substitution in equation 3 with $a = -9.8$, giving

$$v^2 = 400 + 441,$$

giving the speed as 29 ms^{-1}.

<u>Example 4.11</u>

A particle is projected vertically upwards with a speed of 14 ms^{-1}. Find the maximum height reached and the time taken before the particle returns to the point of projection.

The maximum height is given from equation 5 as 196/19.6 = 10 m, and the displacement s m from the point of projection at time t s is $s = 14t - 4.9t^2$. It returns to the projection point when $s = 0$, that is, after $\frac{14}{4.9} = 2.86$ s. The other solution corresponding to $s = 0$ is $t = 0$, which represents the starting time.

<u>Exercises 4.2</u>

Take $g = 9.8$ ms^{-2} throughout this exercise. Questions 1 to 5 refer to a particle projected vertically upwards from a point O with speed u ms^{-1}.

1 $u = 20$; find the height reached and the time taken to reach the ground again.

2 The maximum height reached is 45 m. Find u and the time taken to first reach a height of 40 m above O.

3 $u = 45$; find the times at which (a) the particle speed is 35 ms^{-1} (b) the particle is at a height 90 m above O.

4 It is found that the two times at which the particle is 35 m above O differ by 6 s. Find u.

5 $u = 25$; find the time taken for the particle to reach 30 m below O.

Questions 6 and 7 refer to a particle projected vertically downwards from O with speed u ms^{-1}.

6 $u = 5$; find the speed when the particle has dropped 10 m and the time taken to reach this position.

7 $u = 4$; find the distance fallen in 9 s.

8 A particle is projected upwards from a point O with speed u, and at time T later, a second particle is projected upwards from O with the same speed. Find the time that passes after the initial projection before they meet.

9 A ball is thrown vertically upwards with a speed of 12 ms^{-1} from a point at a height of 2 m above the ground. Find the speed with which it reaches the ground. Given that the ball bounces directly upwards with half the speed with which it hits the floor, find the height to which it first bounces.

10 A ball thrown vertically upwards with a speed of 20 ms^{-1} hits the ground 5 seconds later. Find the height above the ground from which it was thrown.

11 A ball is dropped from the top of a building of height 24 m and, at the same instant, another ball is thrown vertically upwards from the ground so as to hit the other ball. The initial speed of this second ball is 12 ms^{-1}. Find

(i) the time when the balls collide,

(ii) the height at which they collide.

12 A stone is dropped from the top of a high building. The stone takes 1 s to drop from the ninth floor to the eighth and 0.5 s to fall from the eighth floor to the seventh. Find the distance between the floors, assuming that it is constant.

4.4 Proofs of some basic results

If the acceleration has a constant value a, then the definition of acceleration as a derivative gives

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a\,.$$

This equation can be integrated to give

$$v = at + \text{constant}.$$

If the velocity at $t = 0$ is u, then substituting $t = 0$ in the above gives the constant as u, so that $$v = u + at, \;.$$

which is equation 1.

Using the definition of velocity as a derivative, this equation can be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u + at.$$

This equation can be integrated to give

$$x = ut + \frac{1}{2}at^2 + \text{constant}.$$

If x is measured from the position when $t = 0$, and denoted by s, then substitution in the above equation gives the constant to be zero and

$$s = ut + \frac{1}{2}at^2,$$

which is equation 2.

Squaring equation 1 gives

$$v^2 = u^2 + 2uat + a^2t^2,$$

and substituting into this equation from equation 2 gives equation 3.

Also equation 2 can be rearranged as

$$s = \frac{1}{2}(2ut + at^2) = \frac{1}{2}t(u + u + at) = \frac{1}{2}t(u + v),$$

which is equation 4.

The above derivation is the simplest type of example that you can get of solving differential equations. One important point to notice is that every integration brings in an arbitrary constant and each constant has to be found from initial conditions.

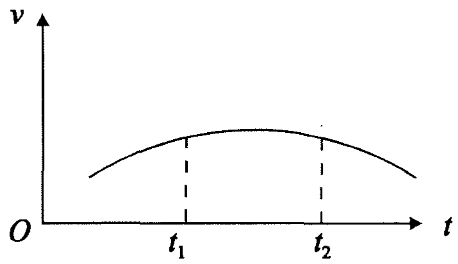
Integrating the equation

$$v = \frac{\mathrm{d}x}{\mathrm{d}t},$$

between two values of t (t_1 and t_2, for example) gives

$$[x]_{\text{at } t = t_2} - [x]_{\text{at } t = t_1} = \int_{t_1}^{t_2} v\,\mathrm{d}t$$

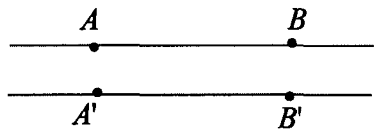
The left hand side is the distance between the two points corresponding to $t = t_2$ and $t = t_1$ and the right hand side is the area under the v - t diagram between $t = t_2$ and $t = t_1$, as shown in the diagram below.



This confirms the statement made in 4.2 that the area under the v - t diagram corresponds to the total distance travelled.

Miscellaneous Exercises 4

1



The diagram shows two parallel tracks for toy cars, the segments AB and $A'B'$ are each of length 8.2 m. At time $t = 0$, a toy car P passes through A with speed 0.8 ms^{-1} and moves towards B with constant acceleration of 0.3 ms^{-2} in the sense from A to B. At time $t = 0$, a toy car Q passes through B' with speed 0.4 ms^{-1} and moves towards A' with constant acceleration of 0.1 ms^{-2} in the sense from B' to A'. Modelling the cars as particles, find

(a) the speed of P when $t = 1$,

(b) the distance of P from A when $t = 1$,
(c) an expression for PQ at time t s,
(d) the times when P and Q are at a distance of 2.8 m apart.
(In parts (c) and (d), the perpendicular distance between the tracks is to be neglected.)

2 A particle P is projected along rough horizontal ground from a fixed point A towards a second fixed point B, where $AB = 60$ m, with speed 4 ms^{-1} and moves under a constant frictional retardation of $\frac{1}{3}$ ms^{-2}. At the same instant, a particle Q is projected along the ground from B towards A with speed v ms^{-1} and is subject to a constant frictional retardation of 2 ms^{-2}. Show that P will come to rest after 12 s at a distance of 24 m from A. Hence deduce that the particles collide if $v \geq 12$.
Show that for the case $v = 13$, the collision occurs after 6 s.
Find the value of v such that the collision occurs after 4 s and determine the distance from A at which the collision takes place.

3 A car passes a point A with speed 35 ms^{-1} and continues at the same speed. Two seconds later a police motor-cyclist sets off from A. He accelerates at a rate of 6 ms^{-2} until he reaches a speed of 45 ms^{-1} and then moves at this speed until he's level with the car. Find the distance from A to the point where the motor-cycle and car are level.

4 A cheetah is estimated to be able to run at a maximum speed of 100 kmh^{-1} whilst an antelope can run at a maximum speed of 65 kmh^{-1}. A cheetah at rest sees an antelope at rest, and starts running towards it. The antelope immediately starts moving away. Both animals are assumed to move with constant acceleration and reach their maximum speeds in 4 s. Assuming that both run along the same straight line and that the cheetah catches up with the antelope in 15 s find the distance between the animals when they first started moving.

5 A cage goes down a mine shaft 480 m deep in 45 s. The cage is accelerated uniformly from rest for the first 160 m, it then travels at a constant speed of v ms^{-1} for the next 240 m; finally it is brought to rest with a uniform retardation over the last part of its journey. Find
(i) v,
(ii) the acceleration of the cage on the first part of its journey.

6 A train starting from rest moves with constant acceleration during the first 50 seconds of its journey when it covers 500 m. It then travels at constant speed until it is brought to rest by applying a constant retardation. The distance travelled whilst the retardation is applied is 800 m. By drawing a speed-time graph, or otherwise, find
(i) the maximum speed attained by the train,
(ii) the magnitude of the retardation.

Given that the total journey time was 4 minutes, determine the distance covered at constant speed.

7 A particle moves with constant acceleration from A to B, where $AB = 96$ m, and then moves with constant retardation from B to C, where $BC = 30$ m. The speeds of the particle at A and B are 6 ms^{-1} and u ms^{-1}, respectively, and it comes to rest at C. Find, in terms of u, the times taken by the particle to move from A to B and from B to C.
Given that the total time taken by the particle to move from A to C is 18 seconds, find
(i) the value of u,
(ii) the acceleration and the retardation of the particle.

8 A train driver, in order to satisfy speed restrictions on a length of track, has to apply a constant retardation of 1 ms^{-2} in order to reduce the speed from 40 ms^{-1} at a point A to a speed of 10 ms^{-1} at another point B. The train travels from B to C, a distance of 3.5 km, at a constant speed of 10 ms^{-1} and then moves with constant acceleration of 0.2 ms^{-2} so that its speed at a point D is 40 ms^{-1}. Sketch the velocity-time graph for the journey from A to D, and show that the distance from A to D is 8 km.
Show that the journey from A to D takes 330 s more than it would if the train travelled at a constant speed of 40 ms^{-1} from A to D.

9 A lift travels vertically a distance of 22 m from rest at the basement to rest at the top floor. Initially the lift moves with constant acceleration for a distance of 5m; it then continues with constant speed u ms^{-1} for 14m, a constant retardation is then applied so that the lift comes to rest at the top floor. Find, in terms of u, the times taken by the lift to cover the three stages of its journey.
Given that the total time that the lift is moving is 6 seconds, find
(i) the value of u,
(ii) the values of the acceleration and retardation.

10 A car starting from rest at a point A moves along a straight road with constant acceleration f until it reaches a speed v; it then continues at this speed. When the car starts, a second car is at a distance b behind the first car and moving in the same direction with constant speed u. Find the distance between the cars at time t after the first car has left A
(i) for $0 < t < v/f$,
(ii) for $t > v/f$.
Show that the second car cannot overtake the first one during the period $0 < t < v/f$ unless $u^2 > 2fb$.
Find the least distance between the two cars in the case $u^2 < 2fb$ and $u < v$. State briefly what will happen if $u^2 < 2fb$ and $u > v$.

11 A tube train travels between the two stations at a distance of 2.7 km apart. The train either moves with constant acceleration of 0.25 ms^{-2} or stays at a constant speed or moves with constant retardation of 0.5 ms^{-2}. The train is subject to a maximum speed limit of 15 ms^{-1}. Given that the train completes the journey, from rest to rest, in the shortest possible time without exceeding the speed limit, find the time taken and show that the distance covered at maximum speed is 2.025 km.

12 A railway timetabler has to determine the time taken between two stations a distance of 22.5 km apart. In order to do this he assumes that a train, on leaving one station, accelerates at a constant rate for 75 s until it reaches a constant speed of 30 ms^{-1} at a point A. It then continues at this speed to point B when a constant retardation is applied for 75 s so that the train comes to rest in the second station .

(a) Draw a sketch of the velocity-time graph showing the motion of the train.

(b) Find the magnitude of the acceleration and the total distance travelled whilst the train is accelerating.

(c) Find the total time estimated by the timetabler for the journey.

(d) The timetabler realises that in making a timetable he has to insert a safety margin by allowing for the train having to stop at a signal. He does this by assuming that some time after passing A the train is forced to start slowing down to rest and then to stop for 60 s. He assumes that the retardation and acceleration applied are the same as those applied at the start and end of the journey. He also assumes that the point where the train starts slowing down is such that the train will have resumed the constant speed of 30 ms^{-1} before reaching B. Find the total time that would now be estimated for the journey.

13 A motorist travelling at 108 kmh^{-1} sees an obstruction 100 m ahead. After a delay of 0.3 s he applies the brakes and immediately starts to decelerate at a rate of 5 ms^{-2}. How far from the obstruction does he stop?

14 When a car driver sees an obstruction ahead, there is a delay of T s (the driver's reaction-time) before he takes any action. The brakes are then applied so that the car moves with constant retardation f ms^{-2}. When the driver sees an obstruction when travelling at a uniform speed of 12 ms^{-1}, he can bring the car to rest in 20 m. If he is travelling at a uniform speed of 24 ms^{-1} when he sees the obstruction, he can bring the car to rest in 64 m. Find f and T.

The driver sees an obstruction when travelling at 18 ms^{-1} with constant acceleration of 3 ms^{-1}. Show that he can stop the car in 46 m.

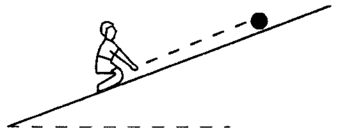
15 A ball is projected vertically upwards with an initial speed of 14.7 ms^{-1}. Find

(i) its greatest height,

(ii) the time taken to reach its greatest height,

(iii) its speed two seconds after projection.

16



The diagram shows a girl rolling a ball up a slope. The slope is such that the ball experiences a retardation of 4 ms^{-2}. Given that the ball starts moving up the slope with speed 2 ms^{-1}, find

(i) the distance it moves up the plane before it comes to instantaneous rest,

(ii) the total time before the ball returns to the girl.

17 At time $t = 0$, a particle A is projected vertically upwards from the point O with speed U. At time $t = U/2g$, a second particle B is projected vertically upwards from O with speed $3U$. Show that the particles collide before A has reached its maximum height.

Find the height above O at which the particles collide.

The particle A is then projected for a second time from O and a third particle C is projected vertically upwards from O at a time $U/2g$ later. This collides with A at time U/g, after A was projected. Find the speed of projection of C.

18 At time $t = 0$, a particle is projected vertically upwards from O with speed 19.6 ms^{-1} and, two seconds later, a second particle is projected from O with the same speed. Express the heights of both particles above O in terms of t and hence, or otherwise, find the value of t when they collide.

Find the speeds of the particles at the instant of collision.

19 Two free-falling raindrops leave the top of a cliff of height h such that the second one begins to fall when the first one has already fallen a distance s. Show that the distance between the drops when the first drop hits the ground is $2\sqrt{sh} - s$.

If the height of the cliff is 28 m and this final distance apart is 3 m, find the value of s, to the nearest centimetre. Take g as 9.8 ms^{-2}.

Chapter 5

Dynamics of Rectilinear Motion

After working through this chapter you should

- know Newton's laws of motion and be able to apply them to determine the rectilinear motion of bodies under the action of given forces,
- be able to solve simple problems involving the motion of connected particles.

5.1 Newton's laws of motion

First law

This law states that every body continues in a state of rest or of uniform motion in a straight line unless compelled to change that state by a force. If a body changes from a state of uniform motion, that body has an acceleration (or retardation) and, therefore, Newton's first law can be interpreted as stating that a force acting on a body produces an acceleration (or retardation) and that any body which possesses an acceleration (or retardation) is being acted upon by a force.

Second law

This second law needs the introduction of a new fundamental quantity, the mass. You will probably have used the idea of mass of a body though you may not have a clear idea of what it means. The theoretical method of defining and measuring mass is given in 5.4. Mass is often defined as the quantity of matter in a body but this is not a particularly good definition. The important point is that the mass of a body is a fundamental property of that body and that the unit of mass can be defined independently of units of time, length or force. The unit used in the S.I. system is the kilogram (kg), and 1000 kg is one tonne. Newton's second law states that the component of a force in a given direction acting on a body is equal to the product of the mass of the body and the acceleration in the given direction. The only problems considered will be those when the acceleration is along a

fixed direction. Motion will not be possible in any other direction, therefore there can be no acceleration in any other direction and therefore the forces in any other direction are in equilibrium. Symbolically

$$F = kma,$$

where a is the acceleration in a reference direction along the line, F is the component of the force along the line in the reference direction and k is a positive constant. It is very important always to remember to choose a particular reference direction at the start of any problem.

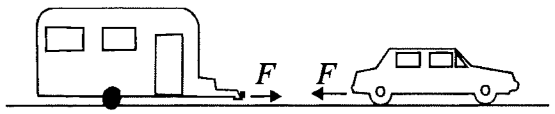
In the S.I. unit system the unit of force (the newton) is chosen so that $k = 1$, so that one newton produces unit acceleration (1 ms^{-2}) when acting on unit mass (1 kg). In this system the above equation becomes

$$F = ma.$$

You will come across problems where not all the forces are in the fixed direction of motion. In this case, as mentioned above, the forces in any other direction are in equilibrium and the equation of motion has to be supplemented by this additional condition.

Third law

Newton's third law states that action and reaction are equal and opposite. You have already met this in Statics but it also applies to bodies in motion. In the diagram, if the car exerts a force to the right and of magnitude F on the caravan, then the caravan exerts a force of magnitude F to the left on the car.



5.2 Motion of single particles under the action of constant forces

It is very important in order to set up a correct equation of motion to make a clear sketch showing the position of the particle at any time and the reference direction and to mark, as in problems in Statics, all the forces acting. The next stage, in most problems, is to find the component of the force in the reference direction and then Newton's second law can be used to find the acceleration. For constant forces, the position of a particle at any time can be found by using the formulae in chapter 4.

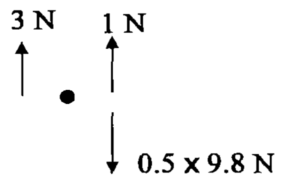
In some problems the direction of motion may not be obvious. This would occur, for example, when there are two or more tugs pulling a ship from different directions. In this case the resultant would have to be found and the magnitude and direction of the acceleration then determined by Newton's second law.

Occasionally, the forces are not known but the motion of a particle can be observed so that the acceleration can be calculated and hence the force determined.

Example 5.1

A stone of mass 0.5 kg is falling through water. The buoyancy of the water provides an upward force of 3 N and the resistance provides a further upward force of 1 N. Find the acceleration of the stone.

The reference direction is taken to be vertically downwards. The vertical forces acting are the force of gravity of 0.5g downwards, the resistance and the buoyancy. These forces are shown in the diagram.



The total force in the downward direction is $(0.5 \times 9.8 - 3 - 1)$ N = 0.9 N and if the downward acceleration is a ms^{-2} then Newton's second law gives

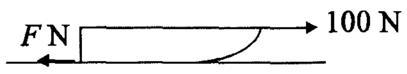
$$0.5\,a = 0.9 \text{ N},$$

and so the acceleration is 1.8 ms^{-2}.

Example 5.2

A toboggan of mass 60 kg is being pulled on horizontal snow by a boy exerting a horizontal force of 100 N and it moves with acceleration 1.5 ms^{-2}. Find the frictional resistance to the toboggan.

The reference direction is chosen to be that in which the boy is pulling. The only other force acting on the toboggan is the frictional resistance F N, which will be in the opposite direction to that in which the boy is pulling. The forces are shown in the diagram below.



The total force to the right is, therefore, $(100 - F)$ N. Therefore Newton's law gives

$$100 - F = 60 \times 1.5,$$

so that the resistance is 10 N.

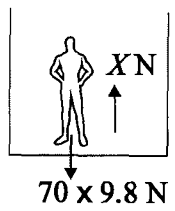
Example 5.3

A man of mass 70 kg is in a lift. Find the forces exerted by the floor of the lift on the man

(i) when the lift descends with an acceleration of 0.3 ms^{-2},

(ii) when on its downward journey, the lift is slowing with a retardation of 0.3 ms^{-2},

(iii) when the lift is descending with acceleration g.



The reference direction is taken vertically downwards, and the forces are as shown in the diagram.

(i) The only forces acting on the man are the force of gravity down and the unknown reaction of the floor, of magnitude X N, acting up. The total downward force in newtons is therefore $(70 \times 9.8 - X)$ N. Therefore

$$70 \times 0.3 = 70 \times 9.8 - X.$$

Hence the force is 665 N.

So the force exerted by the floor on the man is 665 N which is less than the man's weight of 70×9.8 (= 686) N. This would be demonstrated in practice by a slight lightening effect on the legs of the man when the lift starts to accelerate downwards. If the man had been standing on bathroom scales in the lift, then the force would be the reaction of the scales on the man and this, as discussed in 2.4, would be the reading on the scales and therefore his weight would have apparently been changed.

(ii) A retardation of 0.3 ms^{-2} means that the downward acceleration is $-$ 0.3 ms^{-2} and, therefore, still taking the downward direction to be the reference direction, with the upward force exerted on the man now being Y N, Newton's second law gives

$$-70 \times 0.3 = 70 \times 9.8 - Y,$$

the reaction is now 707 N and there is an extra thrust on the legs as the lift slows down when going downwards. This means that the man's apparent weight would have increased.

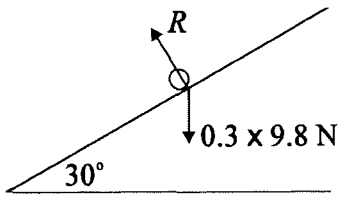
(iii) Replacing 0.3 by 9.8 in the equation of motion in (i) gives

$$70 \times 9.8 = 70 \times 9.8 - X.$$

This gives $X = 0$ so that the man would appear weightless! Effectively the lift would be in 'free fall 'and this would only occur if the cable broke.

Example 5.4

A particle of mass 0.3 kg is projected up a line of greatest slope of a smooth plane inclined at an angle of 30° to the horizontal. Given that its initial speed is 19.6 ms^{-1}, find how far up the plane it travels before coming to instantaneous rest.



The forces acting on the particle are shown in the diagram and this is a simple example of the situation when there are forces acting in directions other than that of the motion. The force due to gravity has a component $0.3 \times 9.8 \times \sin 30^\circ$ N acting down the plane and one $0.3 \times 9.8 \times \cos 30^\circ$ N perpendicular to the plane. The forces off the line of greatest slope are in equilibrium and therefore the reaction R of the plane is equal to $0.3 \times 9.8 \times \cos 30^\circ$ N. Taking the reference direction for acceleration up the plane, Newton's law shows that the acceleration, a ms^{-2}, satisfies

$$0.3\,a \;=\; -\,0.3 \times 9.8 \times \sin 30^\circ,$$

so that the acceleration is – 4.9 ms^{-2}, i.e. it is a retardation.

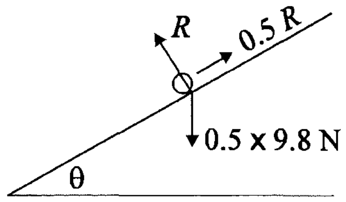
The distance up the plane can now be found from equation 3 of Chapter 4 ($v^2 = u^2 + 2as$) so that

$$0 \;=\; 19.6^2 - 9.8s,$$

showing that the distance travelled up the plane before the particle comes to instantaneous rest is 39.2 m.

Example 5.5

A particle of mass 0.5 kg is sliding down a rough plane inclined at an angle θ to the horizontal where $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$. The coefficient of (sliding) friction between the plane and the particle is 0.5. Find the acceleration of the particle.



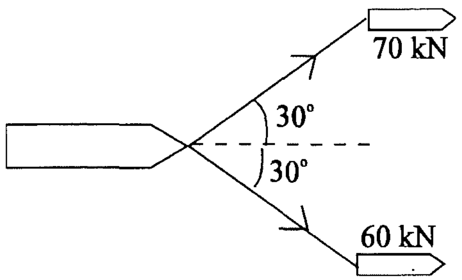
The forces acting are shown in the diagram. Since the particle is sliding down the plane, the friction force will be up the plane and of magnitude 0.5 R, where R denotes the normal reaction of the plane. The force of gravity has a component down the plane of magnitude $0.5 \times \sin\theta \times 9.8$ N and $0.5 \times \cos\theta \times 9.8$ N perpendicular to the plane. As in the previous example, the forces not along the plane are in equilibrium so the reaction of the plane is $0.5 \times \cos\theta \times 9.8$ N = 2.94 N. This gives the friction force as 1.47 N. The acceleration a ms^{-2} down the plane therefore satisfies

$$0.5\,a = \;0.5 \times \sin\;\theta \times 9.8 - 1.47,$$

giving the acceleration down the plane as 4.9 ms^{-2}.

Example 5.6

The diagram shows two tugs towing a tanker of mass 20 000 tonnes, the direction and magnitude of the forces exerted by the tugs are shown. Find the magnitude and direction of the acceleration of the tanker.



The first step is determining the resultant of the forces. This has a component of $(60 \cos 30^\circ + 70 \cos 30^\circ)$ kN = 112.58 kN along the dotted line and

$(70 \sin 30^\circ - 60 \sin 30^\circ)$ kN = 5 kN perpendicular to the dotted line and up the page. The resultant is therefore of magnitude $\sqrt{112.58^2 + 5^2}$ kN = 112.7 kN at an angle θ above the dotted line where $\tan\theta = \frac{5}{112.58} = .044$.

Therefore θ is 2.5° and the acceleration is of magnitude $\frac{112.7}{20}$ ms^{-2} = 5.64 ms^{-2} at an angle of 2.5° above the dotted line.

Exercises 5.1

1 A tug tows a barge of mass 9000 kg with acceleration 0.1 ms^{-2}. The water resistance to the movement of the barge is 900 N. Given that the tow rope is horizontal, calculate the tension in this rope.

2 The resistance to the motion of a car of mass 700 kg is 400 N and the driving force exerted by the engine is 1800 N. Find the acceleration of the car.

3 A constant retarding force of 2000 N is applied to a car of mass 800 kg moving on a level road so as to reduce its speed from 20 ms^{-1} to 10 ms^{-1}. Find, for the interval during which the car reduces speed,

(i) the time taken,

(ii) the distance covered.

4 A body of mass 10 kg is moved vertically upwards from rest by a constant force to a height of 5 m above its starting point and given a speed of 6 ms^{-1}. Find the magnitude of the force.

5 The resistance to the motion of a car moving along a horizontal road is 1500 N. The engine of the car exerts a constant force of 5400 N and the car reaches a speed of 24 ms^{-1} from rest in 8 s. Find

(i) the acceleration of the car during the first 8 s,

(ii) the mass of the car,

(iii) the distance travelled by the car during the first 8 s.

6 A child of mass 28 kg stands in a lift moving with an upwards acceleration of 1.4 ms^{-2}. Find

(i) the force exerted on the child by the floor of the lift,

(ii) the force exerted on the floor by the child.

7 A lift accelerates upwards from rest at a constant rate of 0.5 ms^{-2} to a maximum speed of 2 ms^{-1} and after moving at this constant speed it moves with constant retardation of 0.25 ms^{-2} until it comes to rest.

A parcel of mass 10 kg is standing on the floor of the lift during this ascent from rest to rest. Find the magnitude of the force exerted by the floor of the lift on the parcel during each stage of the ascent.

8 A man of mass 180 kg stands on bathroom scales on the floor of a lift. The lift is moving with a downwards acceleration of 2 ms^{-2}. What is the reading on the scales?
The man then pushes on the ceiling of the lift with a walking stick. Which of the following readings is likely to be the correct one 1420 N or 1390 N?
The man then applies the same force to the walking stick with the latter pushing the floor of the lift. Find the reading of the scales in this case.

9 A car is moving along a horizontal road at a steady speed of 30 ms^{-1}. Find the ratio of the magnitude of the constant force necessary to bring the car to rest in a distance of 40 m to the magnitude of the constant force necessary to bring the car to rest in a time of 8 s.

10 A stone is sliding over the ice on a pond and it slides a distance of 200m before coming to rest from a speed of 15 ms^{-1}. Find the coefficient of friction.

11 A ring of mass 0.01 kg is threaded on a smooth fixed vertical wire. A force of 0.4 N acts on the ring and the line of action of the force makes an angle of 60° with the upward vertical. Find the magnitude of the upward acceleration of the ring.

12 A particle of mass m moves up a line of greatest slope of a plane inclined at an angle θ to the horizontal, where $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$, under the action of a force F of magnitude $2mg$ acting parallel to the plane. Given that the coefficient of friction between the particle and the plane is 0.5, find the acceleration of the particle.

13 A particle of mass m is projected up a line of greatest slope of a plane inclined at an angle θ to the horizontal, where $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$, with speed 14.7 ms^{-1}.

(i) Find, for the case when the plane is smooth, how far up the line of greatest slope the particle will travel.

(ii) Find, for the case when the plane is rough and the particle travels a distance of 14 m up the line of greatest slope before coming to rest, the coefficient of friction between the particle and the plane.

14 A particle of mass 0.2 kg is pushed up a line of greatest slope of a plane which is inclined at an angle θ to the horizontal, where $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$, by a horizontal force of magnitude 2 N. Given that the plane is smooth, find the magnitude of the acceleration of the particle.

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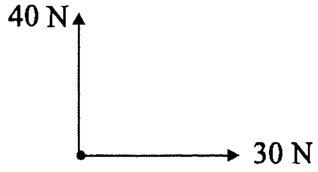


The diagram shows a particle of mass 0.5 kg on a smooth floor being pulled in two perpendicular directions by forces of magnitude 30 N and 40 N respectively. Find the direction in which the particle moves and the magnitude of its acceleration.

5.3 Motion of connected particles under the action of constant forces

Motion along a single line

The simplest type of problems with two, or more, particles connected together are those where all the particles are moving along the same line e.g. a car pulling a trailer, and therefore all particles have the same acceleration. In more complicated cases, the particles are connected by a string passing over a pulley so that they are not all moving in the same straight line.

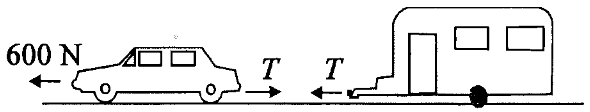
The general approach is still the same. Draw a diagram showing the reference direction and the forces acting on each particle. In these case it is necessary to include the forces between the particles and to remember that Newton's third law has to be satisfied. It is often necessary, in order to solve a particular problem, to apply Newton's second law to both particles or, alternatively, to one particle and to the system as a whole. This avoids calculating the force between the particles. If you consider just one of the particles, then you have to include the force between the particles.

Example 5.7

A car of mass 900 kg tows a caravan of mass 600 kg along a horizontal road. The driving force due to the engine is 600 N. Given that there are no resistances acting, find

(i) the acceleration of the car and caravan,

(ii) the tension in the tow bar.



The forces acting are shown in the diagram. If the system of car and caravan is taken together, the tension forces in the tow bar cancel and the total force acting is 600 N. Newton's second law gives, denoting the acceleration by a ms^{-2},

$$1500\,a = 600 \text{ N},$$

so that the acceleration is 0.4 ms^{-2}.

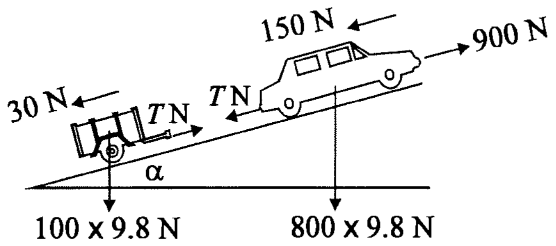
Newton's second law can now be applied to the caravan. The only force acting is the tension T, therefore

$$T = 600 \times 0.4 \text{ N} = 240 \text{ N}.$$

It would have been possible to use the equation of motion of the car. The force acting is $(600 - T)$ N and this is equal to 900×0.4 N, giving the same answer.

Example 5.8

A car of mass 800 kg is towing a trailer of mass 100 kg up a hill inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{14}$. The driving force due to the engine is 900 N and the resistances to the car and trailer are, respectively, 150 N and 30 N. Find the acceleration of the car and the tension in the tow bar.



The forces acting are shown in the diagram. In this case the component of the weight down the hill has to be taken into account. Taking the car and trailer together, the total resistance acting down the hill is 180 N. The component of the total weight down the hill is $900 \times 9.8 \times \sin\alpha$ N = 630 N. The total component of force, acting up the hill, is therefore $(900 - 630 - 180)$ N = 90 N. The acceleration, a ms^{-2}, is given by

$$900a = 90,$$

so that

$$a = 0.1.$$

Considering next the trailer. The component of the weight down the hill is

$$100 \times 9.8 \times \sin\alpha \text{ N} = 70 \text{ N},$$

the resistance is 30 N and therefore the tension , T N, satisfies

$$T - 30 - 70 = 100 \times 0.1.$$

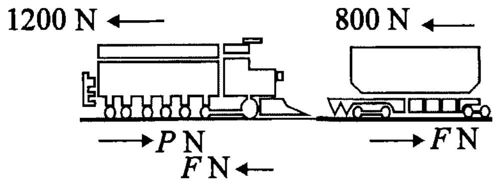
so that the tension is 110 N.

Example 5.9

An engine of mass 50 tonnes pushes a carriage of mass 10 tonnes with an acceleration of 0.4 ms^{-2}. The resistance to the motion of the engine is 1200 N and to the motion of the carriage is 800 N. Find

(i) the total driving force of the engine,

(ii) the force exerted on the carriage by the engine.



The total force on the two is (P – 2000) N to the right, where P N denotes the driving force, and therefore, by Newton's second law,

$$P - 2000 = 60000 \times 0.4,$$

so that the driving force is 26 000 N.

The force acting to the right on the carriage is (F – 800) N, where F N is the force exerted by the engine. Therefore

$$F - 800 = 10000 \times 0.4,$$

so that the force exerted by the engine is 4800 N.

Exercises 5.2

1 A car of mass 700 kg tows a caravan of mass 500 kg along a horizontal road. Given that the driving force is 300 N and neglecting resistances, find the acceleration of the car.

2 A car of mass 900 kg is pulling a caravan of mass 600 kg, by means of a tow bar, along a straight horizontal road. The resistive forces opposing the motions of the car and the caravan are 120 N and 60 N respectively. Given that the car is accelerating at 2 ms^{-2}, find

(i) the tension in the tow bar,

(ii) the force being produced by the engine of the car.

3 A car of mass 800 kg tows a caravan of mass 400 kg against a total resistance of 600 N. Given that the acceleration of the car is 0.8 ms^{-2} and that the resistances on the car and caravan are proportional to their masses, find

(i) the driving force,

(ii) the tension in the tow bar.

4 A car of mass 800 kg has a driving force of 2.1 kN when pulling a caravan of mass 400 kg and has a constant acceleration of 0.5 ms^{-2}. Given that the resistances on the car and caravan are proportional to their masses, find these resistances and the tension in the tow bar.

5 A tug of mass 50 000 kg tows three barges in line behind it along a canal. Each barge is of mass 10 000 kg and the resistance to the motion of each barge is 800 N. The resistance to the motion of the tug is 3000 N. Calculate the tensions in the tow ropes between

(i) the tug and the first barge,

(ii) the first and second barges,

(iii) the second and third barges, when

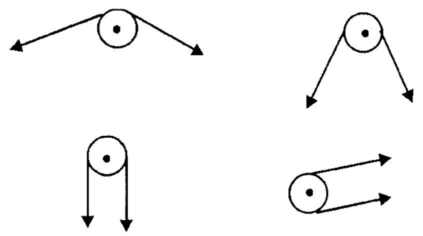
(a) the tug and barges are moving at a uniform speed,

(b) they are all accelerating at 0.5 ms^{-2}.

6 A car of mass 1200 kg is pulling a caravan of mass 300 kg up a slope inclined at an angle α to the horizontal, where $\sin \alpha = \frac{1}{196}$. The driving force is 1800 N and the resistances to the car and caravan are 275 N and 100 N respectively. Find the acceleration of the system and the tension in the tow bar.

Motion involving pulleys

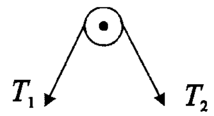
In some problems involving connected particles each particle is moving on a straight line but they are not moving in the same straight line. This happens when the connection (e.g a string) passes over a pulley. Some typical configurations are shown in the diagram.



In this case, the acceleration of the particles (assuming the string is inextensible) has the same magnitude but not the same direction. The same approach has again to be used i.e. showing all the forces on a clear diagram and applying Newton's law separately to each particle.

Modelling a pulley

A pulley is effectively a device for changing the direction of a force and is normally free to turn about an axis. If the rim of the pulley is smooth then a string passed over it simply slides along it without making the pulley turn and the pulley is then effectively the same as a smooth peg. This point was discussed in the Glossary in 2.2. In practice, however, the rim of the pulley is rough so that normally a string does not slide relative to it and the string makes the pulley rotate.

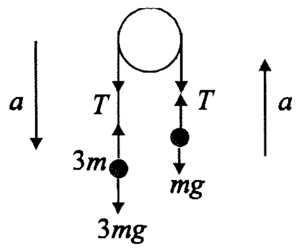


It is possible to show that the difference in the tensions T_1 and T_2 at the points shown in the diagram is proportional to the mass and radius of the pulley and if these are neglected (the pulley is said to be small) then the tensions are equal. This is the normal assumption. Since the pulley can rotate, then there may be friction at the axis and this also can make T_1 different from T_2. If this friction can be neglected, the pulley is described as smooth, which is slightly different from the use in Statics. To summarise, in Dynamics a string is assumed not to slide relative to the pulley and a small smooth pulley is one where the tensions T_1 and T_2, shown above, are equal and there is no frictional effect at the axle of the pulley.

__Example 5.10__

A light inextensible string passes over a small smooth pulley and particles of masses $3m$ and m, are attached to the ends of the string and can move, with the parts of the string not in contact with the pulley being taut and vertical. Find the acceleration of the particles and the tension in the string.

Since the string is light, the tension will be constant throughout any straight length, and the fact that the pulley is small and smooth ensures the tension is the same throughout the string.



The forces acting on the particles are shown in the diagram; since the string is taut, the particles will both have accelerations of the same magnitude a but in opposite directions,

as shown in the diagram. The downward component of the force on the heavier particle is $3mg - T$ so that applying Newton's second law to it gives

$$3mg - T = 3ma.$$

The acceleration of the lighter particle is upwards and the upwards component of the force on it is $T - mg$. Newton's law gives

$$T - mg = ma.$$

Adding the equations gives

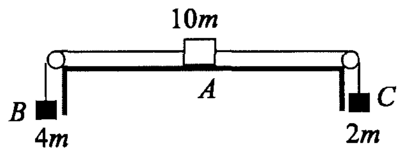
$$2mg = 4ma,$$

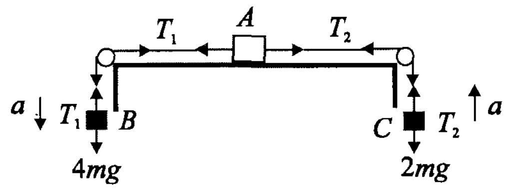
so that $a = \frac{g}{2}$. Substituting this into either of the other equations gives $T = \frac{3mg}{2}$.

It is worth noticing that the equation for a is precisely the equation that would have been obtained by considering the horizontal motion of the system formed by rotating clockwise one of the vertical parts of the string and the other part counterclockwise, so that the string becomes straight and horizontal with there being no change, relative to the string, in the forces acting. In this changed problem, the accelerations of both particles are in the same direction but the forces are in opposite directions. It is certainly not a correct method to use for this problem and others involving strings over pulleys and **it should not be used in examinations** since it needs to be proved correct in each case. It is also very easy to make a mistake and therefore marks, even when awardable, might be lost. Also most examination questions tend to ask for the tension and this needs the writing down of at least one equation of motion. There is therefore very little benefit in attempting to use this particular 'shortcut'. At best it should only be used as a check on the algebra involved in eliminating the tension.

__Example 5.11__

The left hand diagram below shows a particle A, which is of mass $10m$ and lies on a smooth rectangular table. Particle A is connected by light inextensible strings to two particles B and C of masses $4m$ and $2m$, respectively, and passing over smooth pulleys at the opposite edges of the table. The strings are both perpendicular to the edges of the table. Find the acceleration of A, B and C and the tension in the string attached to C.





The right hand diagram shows the forces acting on the particles; there is no reason to assume the tensions in the strings are equal and so they are denoted by T_1 and T_2. If it is assumed that B has an acceleration a downwards, then A has an acceleration of a to the left and C has an acceleration a upwards. The equations of motion of the particles A, B, C are respectively

$$\begin{aligned} 4mg - T_1 &= 4ma, \\ T_1 - T_2 &= 10\,ma, \\ T_2 - 2mg &= 2ma. \end{aligned}$$

Adding these gives $a = \frac{g}{8}$ and substituting in the last equation gives $T_2 = 2.25\,mg$.

Example 5.12

Find the acceleration for the configuration of Example 5.11 when the table is rough with coefficient of sliding friction 0.1.

The only difference is that there will be a frictional force acting on A. The reaction of the table is 10 mg and the friction force is therefore mg and it acts to the right since A moves to the left. The middle equation then becomes

$$T_1 - T_2 - mg = 10\,ma,$$

adding the equations now gives $a = \frac{g}{16}$.

Exercises 5.3

Questions 1 to 3 refer to a light inextensible string passing over a small smooth pulley, with particles of masses m_1 and m_2 attached one at each end of the string. Find, in each case, the magnitude of the acceleration of the particles and the tension in the string.

1 $m_1 = 5$ kg, $m_2 = 3$ kg.

2 $m_1 = 7$ kg, $m_2 = 3$ kg.

3 $m_1 = 4M$, $m_2 = 6M$.

Questions 4 to 6 refer to a particle A of mass m_1 on a horizontal plane and attached by a light inextensible string, which passes over a small smooth pulley at the edge of the plane, to a particle B of mass m_2 which hangs freely with the string vertical. The vertical plane through B and the pulley contains that part of the string between A and the pulley.

4 The plane is smooth, $m_1 = 3$ kg, $m_2 = 5$ kg. Find the magnitude of the acceleration of the particles and the tension in the string.

5 The plane is smooth, $m_1 = 1$ kg, $m_2 = 4$ kg. Find the magnitude of the force exerted on the pulley.

6 The plane is rough with coefficient of friction 0.5, $m_1 = 3$ kg, $m_2 = 6$ kg. Find the tension in the string.

Questions 7 to 10 refer to a particle A of mass m_1 on a plane inclined at an angle α to the horizontal. It is connected by a light inextensible string, which passes over a small smooth pulley at B and which hangs freely with the string BC vertical, to a particle C of mass m_2. The portion AB of the string is parallel to a line of greatest slope of the plane.

7 The plane is smooth with $\alpha = 30°$, $m_1 = 1$ kg, $m_2 = 4$ kg. Find the magnitude of the acceleration of the particles.

8 The plane is smooth with $\sin\alpha = \frac{4}{5}$, $m_1 = 5$ kg, $m_2 = 3$ kg. Find the tension in the string.

9 The plane is rough with coefficient of friction 0.25, $\sin\alpha = \frac{3}{5}$, $m_1 = 2$ kg, $m_2 = 8$ kg. Find the magnitude of the acceleration of the particles.

10 The plane is rough with coefficient of friction 0.25, $\sin\alpha = \frac{5}{13}$, $m_1 = 13$ kg, $m_2 = 15$ kg. Find the magnitude of the acceleration of the particles.

5.4 The concept of mass

As mentioned in 5.1, the mass of a body is a fundamental property of a body and the unit of mass can be chosen independently of those of length and time. In this section, the idea of mass and its measurement is described in slightly more detail.

If, for example, a small body is suspended from a light spring and set in motion vertically then, in principle, its acceleration can be measured for various extensions of the spring. If this process is repeated with a different body then its acceleration for various extensions can also be measured and this can be done for many bodies. It would be found that the accelerations at the same extension would be different, the force exerted by the spring, being dependent only on the extension, would be the same in all cases. It would also be found that for all extensions, i.e. for different values of the force acting, the ratios of the magnitudes of the accelerations would be constant for any pair of bodies. Therefore, there exists an independent relative property of the bodies which is demonstrated by this ratio. This property is referred to as the mass of a body.

Having recognised the existence of mass, the next step is to quantify it. The accelerations produced by the same force (i.e. at the same extension) acting on different particles P, Q and R are measured and the magnitude of these accelerations are denoted by a_P, a_Q and

a_R, respectively. The masses of P, Q and R, denoted by m_P, m_Q m_R are then defined so that

$$\frac{m_Q}{m_P} = \frac{a_P}{a_Q}, \quad \frac{m_R}{m_P} = \frac{a_P}{a_R}.$$

If one of these particles (P say) is then chosen to have unit mass, the masses of the other particles can be found by measuring the accelerations. It can also be verified experimentally that, for mass defined in this way, the mass of the combined particle Q and R is the sum of the masses of Q and R.

Therefore, in principle, a method of measuring mass can be established, and the definition of the unit of mass is independent of the choice of units of length, time and force.

Miscellaneous Exercises 5

1 A tug of mass 7000 kg pulls a boat of mass 4000 kg by means of an inextensible horizontal tow rope along a straight canal. The resistive forces opposing the motions of the tug and the boat are 1400 N and 900 N respectively. Find the tension in the rope when the tug is accelerating at 1 ms^{-2} and the driving force exerted by the tug.

2 When a car of mass 1800 kg is moving with speed 20 ms^{-1} on a straight horizontal road, its engine is switched off. The car is brought to rest in a distance of 500 m by a constant retarding force F newtons. Find F and the time taken for the car to come to rest.

3 The speed of a car of mass 600 kg, moving along a level road, is reduced from 18 ms^{-1} to 8 ms^{-1} by a constant retarding force of 1800 N. Find the time taken for the car to reduce speed and the distance travelled.

4 A lift, starting from rest, moves with constant acceleration for 4 s and with constant velocity for the next 8 s. A constant retardation is then applied so that the lift comes to rest in 4 s at a height of 6 m above its starting point. There is a packing case of mass 20 kg on the floor of the lift.

Find (i) the acceleration and retardation of the lift,

(ii) the reaction of the floor of the lift on the case during each stage of the motion.

5 A man of mass 60 kg stands on the floor of a lift which descends with acceleration 0.6 ms^{-2}. Find the total force exerted between the floor of the lift and the man's feet. A man stands on a weighing machine which rests on the floor of a stationary lift, and the dial of the machine shows 75 kg. The lift then moves upwards and the dial then shows a constant value of 80 kg. Find the upward acceleration of the lift.

The lift then starts to slow down and the dial reading changes to a constant value of 70 kilograms. Find the retardation of the lift.

6 A heavy particle is suspended by a spring balance from the ceiling of a lift. When the lift moves up with constant acceleration a ms^{-2}, the balance shows a reading 1.8 kg. When the lift descends with constant acceleration $3a$ ms^{-2}, the balance shows a reading 1.2 kg. Find the mass of the particle and the value of a.

7 A particle of mass m is moving vertically upwards with speed v at the instant it enters a fixed horizontal layer of material of thickness a which resists its motion with a constant force of magnitude R. Find the condition to be satisfied by v so that the particle passes through the layer.

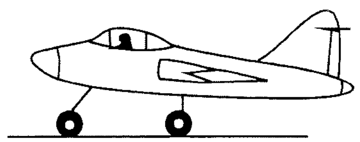
8 A lady of mass 49 kg stands in a lift. Find the reaction of the floor of the lift on the lady when the lift is

(a) moving down with constant acceleration 0.2 ms^{-2},

(b) moving up with acceleration 0.3 ms^{-2}.

Another lady of the same mass stands in the lift when it is stationary. She has to support herself by a walking stick which she presses on the floor of the lift. Assuming that the force exerted by the walking stick on the floor of the lift is vertical, and of magnitude 98N find the reaction of the floor of the lift on the lady.

9(a)

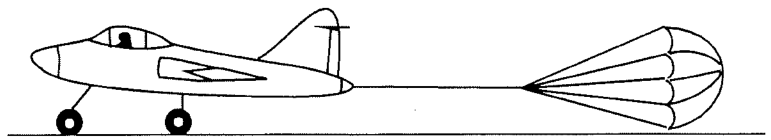


An aeroplane of mass 7000 kg lands at 50 ms^{-1}. It is brought to rest by braking and resistive forces which are assumed to be constant and of total magnitude 7 kN.

(i) Find the retardation of the aeroplane.

(ii) Determine the distance that the aeroplane travels before coming to rest.

(b)



By fitting a tail parachute, with the additional equipment being of mass 200 kg, the distance that the aeroplane travels before coming to rest is reduced by 500 m.

(i) Find, assuming it to be constant, the retardation of the plane when the parachute is used.

(ii) Determine the total drag force in this case.

10 A particle slides with acceleration 3 ms^{-2} down a line of greatest slope of a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. Calculate the coefficient of friction between the particle and the inclined plane.

11 Two particles of mass M and $4M$, respectively, are connected by a light inextensible string passing over a smooth fixed pulley. The particles are released from rest with the hanging parts of the string vertical. Find

(i) the acceleration of the particles,

(ii) the force exerted by the string on the pulley in the subsequent motion.

12 A lift, when empty, is of mass 1200 kg and when it moves upwards with an acceleration of 1 ms^{-2} and carrying N passengers each of mass 80 kg, the tension in the cable is $\frac{1}{2}T$ newtons. The same tension occurs when $N + 5$ people, each of mass 80 kg, are accelerating downwards at 1 ms^{-2}. Find N and T.

13 Two particles of mass $4m$ and m, respectively, are joined by a light inextensible string passing over a smooth peg. The particles are held at rest with the string taut and then released from rest.

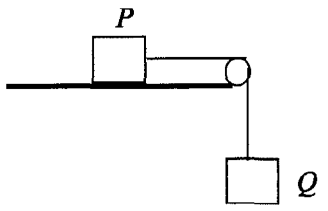
(a) State which of the underlined words enables you to assume that

(i) the tension is constant in the string on either side of the peg,

(ii) the tension is the same on both sides of the peg at the points of contact with the string.

(b) Write down the equation of motion of each particle and determine the acceleration of the particles.

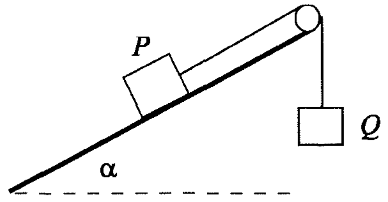
14 The diagram shows a particle P of mass m on a horizontal table and attached to a particle Q of mass $5m$ by a light inextensible string passing over a smooth pulley at the edge of the table. The string is perpendicular to the straight edge of the table and Q can move in a vertical line. The particles are held at rest with the string taut and then released,



(a) Assuming that the table is smooth, show that the speed of Q after it has dropped a distance d from rest is $\sqrt{\frac{5gd}{3}}$.

(b) In an actual experiment with the particles, it is found that the speed of Q after dropping a distance d is less than $\sqrt{\frac{5gd}{3}}$. Suggest one reason for the decrease in speed found by experiment.

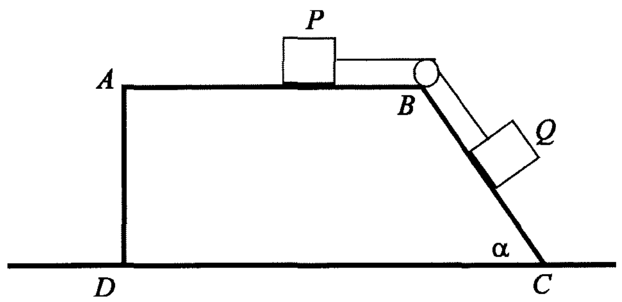
15



The diagram shows a particle P of mass $5m$ on a rough plane inclined at an angle α to the horizontal where $\sin\alpha = \frac{3}{5}$. The coefficient of friction between P and the plane is $\frac{a_P}{a_R}$. A light inextensible string attached to the particle passes parallel to a line of greatest slope of the plane and over a small smooth pulley at the top of the plane. A particle Q of mass $10m$ is attached to the other end of the string and Q can move freely in a vertical line. Given that the particles are released from rest at time $t = 0$, find the tension in the string.

After Q has dropped a distance $10a$ it hits a horizontal plane and then stays at rest on the plane. Find the further distance travelled by P before first coming to instantaneous rest.

16

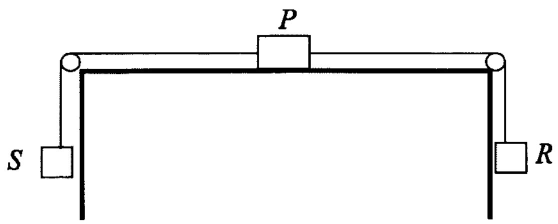


The diagram shows a vertical section $ABCD$ of a block of wood fixed on a horizontal plane. AB is horizontal and BC is inclined at an angle α to the horizontal where $\sin\alpha = \frac{4}{5}$. Particles P and Q, of mass m and $5m$ respectively, are placed on AB and BC and joined together by a light inextensible string passing over a smooth pulley at B. The particles are then released from rest. Find, assuming that P does not reach B and Q does not reach C, the acceleration of the particles and the tension in the string when

(a) AB and BC are smooth,

(b) AB and BC are both equally rough, the coefficient of friction being $\frac{1}{3}$.

17



The diagram shows a particle P of mass $2m$ on a rough horizontal table attached by light inextensible strings to particles R and S of mass $6m$ and $2m$ respectively. The coefficient of friction between P and the table is 0.5. The strings pass over light smooth pulleys on opposite sides of the table so that R and S can move freely with the strings perpendicular to the table edges. Given that the system is released from rest with the strings taut, find the magnitude of the common acceleration of the particles and the tension in the string joining P and S.
After falling a distance d from rest, the particle R strikes the floor which is such that R is brought to rest and then remains at rest. Find the further distance that S rises.

Chapter 6

Impulse and momentum

After reading this chapter you should
- know what is meant by impulse and momentum,
- know and be able to apply the impulse momentum principle,
- know and be able to apply the principle of conservation of momentum to problems of collision of inelastic bodies,
- know Newton's elastic law and be able to apply it, together with the principle of conservation of momentum, to collisions of elastic bodies.

6.1 Impulse momentum principle

There are many examples in everyday life where the velocity of a body is changed fairly quickly. One example is when the brakes of a car are applied very suddenly, another is when a tennis ball is hit. In all these cases a force is acting on a body for a very short period of time. Since the time that the force acts is very short, it is actually very difficult to measure either the time or the force. It turns out however that it is a quantity depending on both the force and the time that actually determines the change in velocity. This quantity is known as the impulse and the idea of impulse is possibly most easily understood by considering the case of a constant force acting for a short period of time. This is not a good model for sharp blows but the basic definition will then be extended to give a more realistic model.
It will be assumed that the component of a force in a given direction is denoted by P and that it acts on a particle of mass m for time T. During this period it is assumed that the velocity of the particle in the given direction changes from u to v. The acceleration due to the force is $\frac{P}{m}$ and applying $v = u + at$ gives

$$v - u = \frac{P}{m}T.$$

This can be rearranged slightly as

$$m(v-u) = PT$$

The right hand side of this equation is directly proportional to the change in velocity and is dependent only on the force and the time for which it acts. This is known as the **impulse of the force** which, for a constant force, is defined as the product of the component of the force in a given direction and the time for which it acts. The unit of impulse is the newton second (Ns).

The product of the mass of a particle and its velocity is known as its **momentum** (the unit of momentum is therefore also the newton second) and therefore the above equation is equivalent to

change in momentum= impulse.

This is the impulse momentum equation; so far it has only been shown to hold for constant forces but it is true whatever force is acting. The momentum, as defined, is actually the linear momentum but as you will not meet any other type of momentum in your course there will be no confusion produced by omitting the adjective linear.

In practice, it is the change of momentum that is observed and the impulse is then determined from the impulse momentum equation. If the time the force is acting is known, then assuming the force is constant, the force could be estimated from the impulse momentum equation.

Suppose that the speed of a car of mass 1000 kg moving with speed 20 ms^{-1} suddenly drops, due to braking, to 15 ms^{-1}. The impulse momentum equation shows that an impulse of magnitude 5000 Ns has been applied but more information would be necessary to find the force producing this change. If the same braking action were carried out (i.e.the same impulse applied) at a speed of 18 ms^{-1}, then the momentum becomes

$$(18000 - 5000)\text{ Ns} = 13000\text{ Ns}$$

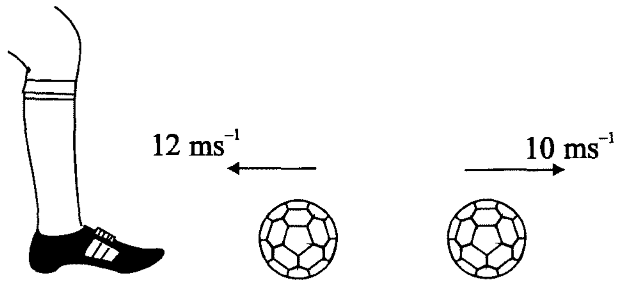
so that the speed drops to 13 ms^{-1}.

Modelling a sharp blow

Although it is unlikely that forces are constant throughout contact, in modelling a sharp blow it is not necessary to consider the precise form of the force but simply say that an "impulse" is applied so that there is an instantaneous change of momentum equal to the impulse. The model is therefore one where it is assumed that there are large forces acting for effectively infinitesimal periods of time so that the impulse is finite.

During any sharp blow there will obviously be forces like gravity acting but, in the limit of zero time, these do not make a contribution to the impulse. This is also true of the tension in an elastic spring or string, neither of which can sustain an impulsive tension.

(The tension is proportional to the extension, the impulsive tension would therefore be the integral of the extension over a zero interval and is therefore zero.) The forces which make a contribution in the limit as time tends to zero are referred to as "impulsive" forces and normally their detailed behaviour is not known. There are no functions, in the normal sense of the word, whose integral over a zero interval is non-zero but it is possible to construct some bounded functions whose integral over a very small interval is finite and non-zero. One such function is $\dfrac{T}{t^2+T^2}$. You will not yet have covered calculating its integral from $t = 0$ to $t = T$, but it can be shown to be equal to $\dfrac{\pi}{2}$. You might like to try and plot it in the range $0 \le t \le T$, for small values of T.



The diagram shows a football, of mass 0.4 kg, moving horizontally with speed 12 ms^{-1} when it is kicked by a footballer so that immediately after leaving him it is moving horizontally with speed 10 ms^{-1}. The momentum of the ball after the kick is 4 Ns to the right and before the kick it is 4.8 Ns to the left. Therefore the change in the momentum to the right is 8.8 Ns and this is the impulse applied by the kick.

There is no way you could find the actual force without making more assumptions about its form but the important thing however is that, even without knowing the precise form of the force, you could now work out what happens when the ball arrives with a velocity of 8 ms^{-1}. The modelling assumption is that the player kicks exactly as before i.e. that the impulse of the force of the kick is 8.8 Ns. The football is assumed to move away with a speed of v ms^{-1}, so that the momentum to the right afterwards is $0.4v$ N s and before it was -3.2 Ns. The impulse momentum principle gives

$$0.4v + 3.2 = 8.8,$$

so that

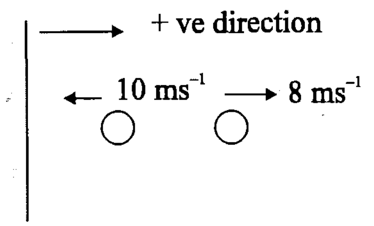
$$v = 14.$$

The impulse associated with a force can obviously be calculated for any force and in fact can be used to solve other problems of motion. It is not a particularly good method for solving other problems and it is really best to just try and keep the idea that impulse is something particularly linked to sharp blows and is proportional to the change of momentum. You should also try and remember that impulse as force × time is a very special case and one which is very unlikely to occur particularly in real problems involving sharp blows.

In solving problems involving impulse and momentum you should remember that, as for problems involving Newton's equations of motion, it is important to pick a reference direction.

Example 6.1

A ball of mass 0.05 kg hits a vertical wall with a speed of 10 ms^{-1}, the direction of motion being perpendicular to the wall. The ball rebounds back with a speed of 8ms^{-1}. Find the impulse exerted by the wall on the ball.



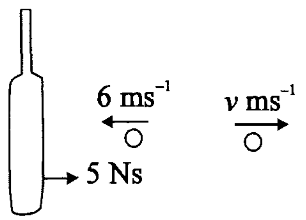
The positive direction is taken to be that away from the wall as shown in the diagram. The linear momentum of the ball immediately after impact is 0.05×8 Ns = 0.4 Ns, the linear momentum immediately before is – 0.05×10 Ns = – 0.5 Ns.

The impulse is therefore 0.4 Ns – (–0.5) Ns = 0.9 Ns.

The impulse is, as you would expect, acting away from the wall. By Newton's third law, there will be an equal and opposite impulse acting on the wall.

Example 6.2

A cricket ball of mass 0.5 kg moving horizontally with speed 6 ms^{-1} is struck by a bat which applies a horizontal impulse of magnitude 5 Ns and directed in the opposite direction to that of the ball. Find the velocity of the ball immediately after impact.



The velocities before and after impact, and the impulse, are assumed to be as shown in the diagram. The momentum before impact is – 0.5×6 Ns and that after is $0.5v$ Ns. Applying the impulse momentum principle gives

$$0.5v + 0.5\times 6 = 5,$$

so that

$$v = 4.$$

Example 6.3

A ball of mass 0.3 kg is moving with speed 5 ms^{-1} just as it hits a horizontal floor and bounces off the floor with a speed of 2 ms^{-1}. Find the impulse exerted on the ball by the floor assuming that the time of contact with the floor may be neglected.

The momentum of the ball immediately after impact is 0.6 Ns upwards and immediately before impact it is –1.5 Ns. The total change in the momentum upwards is 2.1 Ns and this is the impulse applied to the ball. If the time of contact may be neglected, then the impulse due to gravity may be neglected and the impulse exerted by the floor is 2.1 Ns.

Exercises 6.1

1 Find the momentum of the following

(a) a particle of mass 0.03 kg moving with velocity 4 ms^{-1},

(b) a cricket ball of mass 0.5 kg moving with velocity 15 ms^{-1},

(c) a car of mass 1200 kg moving with speed 25 ms^{-1}.

Questions 2 to 4 refer to a particle of mass m kg whose velocity changes from u ms^{-1} to v ms^{-1}. Find the change in momentum.

2 $m = 0.5$, $u = 4$, $v = 6$.

3 $m = 1.6$, $u = 4$, $v = -2$.

4 $m = 2.2$, $u = -3$, $v = -5$.

5 An impulse of magnitude 3.2 Ns is applied to a particle of mass 0.8 kg at rest. The particle is free to move along the x-axis and the impulse is applied in this direction. Find the resulting speed of the particle.

6 A railway truck of mass 1400 kg, moving along a straight horizontal track at 6 ms^{-1}, rebounds from fixed buffers with a speed of 1.5 ms^{-1}.

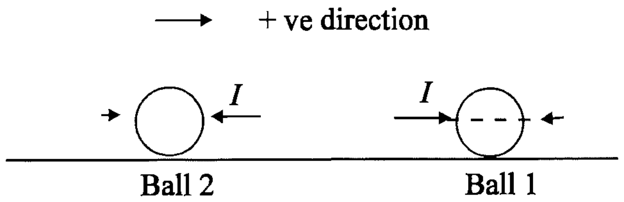
Find the impulse exerted by the buffers on the truck.

7 An ice skater whose total mass is 70 kg receives a horizontal impulse of magnitude 200 Ns when standing at rest. Find the initial speed of the skater.

8 A tennis ball of mass 0.08 kg moving horizontally towards the racquet with speed 6 ms^{-1} is hit by a racket and leaves the racket horizontally with a speed of 12 ms^{-1}. Calculate the magnitude of the impulse on the ball.
9 A cricket ball of mass 0.15 kg moving horizontally with speed 14 ms^{-1} just as it reaches a batsman is hit straight back horizontally with a speed of 24 ms^{-1}. Find the impulse of the bat on the ball.
10 A ball of mass 0.2 kg falls vertically onto a horizontal floor which it strikes with a speed of 10 ms^{-1} and bounces to reach a height of 2.5 m above the floor.
Find the impulse exerted by the floor.

6.2 Conservation of momentum

When a batsman strikes a ball, by Newton's third law, there will, be an opposite impulse acting on the bat, and therefore, on the batsman. This should affect his momentum but normally he would exert a further impulse on the ground to stay at rest. This kind of reaction is particularly obvious in the case of shooting a rifle when there is a recoil. There are however many problems involving sharp blows between bodies where one body does not compensate like a human ball player and motion of both bodies is possible. Obvious examples are snooker balls, or cars colliding. An example of such a problem is that of the collision of two balls moving directly towards each other as in the diagram.



In the collision there will be impulsive forces acting between the balls during the collision. By Newton's third law, the forces exerted by ball 2 on ball 1 during collision are equal and opposite to those exerted by ball 1 on ball 2. Since impulse is the integral of the force, the impulses acting on the balls are equal and opposite. The impulse on ball 1 due to ball 2 is denoted by I and that of ball 2 due to ball 1 is therefore $-I$ (i.e. I to the left).
The positive direction is taken to be towards the right so that by the impulse momentum principle
Change in momentum of ball 1 = I,
Change in momentum of ball 2 = $-I$.

Adding these gives that the change in total momentum of balls 1 and 2 = 0.
Therefore the total momentum of the two balls is unchanged (i.e it is conserved), this is also true for any number of particles and this is the **principle of conservation of momentum** which states that
During any period when there are no external impulses acting on a system of interacting particles the total momentum remains constant.

It is also possible to prove for a system when external impulses are present a **general impulse momentum principle** which states that
If impulses are applied to a system of interacting particles the change in momentum due to the impulse is equal to the sum of the impulses applied.

Three different types of situations involving impulsive motion of a system of particles will be considered
(i) Collisions between bodies which move together after collision; these are called inelastic collisions and can usually be solved by use of the principle of conservation of linear momentum. These problems are considered in detail in 6.3.
(ii) Collisions where the bodies bounce apart after collision; these are called elastic collisions and to solve them it is necessary to know something about the elastic properties of the bodies. The method for solving these problems is given in 6.4.
(iii) Problems where the bodies are connected together by an inelastic string so that they move together. A simple example is two cars connected by a tow rope and one moving off. Apart from the simple problems of two bodies moving along a line, some problems involving particles connected by a string passing over a pulley are also considered.
All these problems are examined in M2.

6.3 Inelastic collisions

Problems such as those when two cars collide with each other are usually quite easy to solve using the principle of conservation of momentum. The total momentum before collision will be known, you have to be very careful to pick a reference direction so that you get the correct signs for the momentum. The method is best seen by working through the following examples.

Example 6.4

A car of mass 1400 kg moving with a speed of 4 ms^{-1} crashes into a stationary car of mass 1000 kg. After collision they move together. Find the common speed immediately after collision.

The reference direction is taken in the direction of the moving car, as shown in the diagram, and the common speed immediately after collision is denoted by v ms^{-1} as shown. The momentum before collision is therefore 1400×4 Ns = 5600 Ns. The momentum after is

$$(1400 + 1000)v \text{ Ns} = 2400v \text{ Ns}.$$

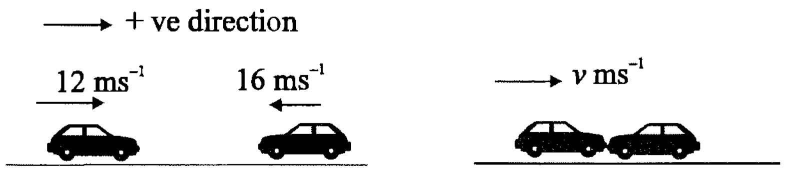
Conservation of momentum gives

$$2400v = 5600,$$

so the cars move together with speed $\frac{7}{3}$ ms^{-1}.

Example 6.5

Two cars of mass 1200 kg and 1300 kg are moving directly towards each other with speeds of 12 ms^{-1} and 16ms^{-1} respectively. After collision the two cars move together. Find their common speed.



The reference direction is taken to be that of the initial motion of the lighter car so that the momentum of the latter is 14400 Ns whilst that of the heavier car is – 20800 Ns, the total momentum before collision is therefore – 6400 Ns.

It is now assumed that both move together with speed v ms^{-1} to the right. The total linear momentum after collision is 2500v Ns and this is equal to – 6400 Ns so v = – 2.56 and therefore both cars move to the left with speed 2.56 ms^{-1}.

Exercises 6.2

Questions 1 to 6 refer to a particle of mass M kg moving with velocity u ms^{-1} colliding with a particle of mass m kg moving with velocity v ms^{-1}. After collision they move together with velocity w ms^{-1}.

1 $M = 0.4$, $m = 0.2$, $u = 3$, $v = 0$, find w.

2 $M = 0.2$, $m = 0.6$, $u = 5$, $v = 0$, find w.

3 $M = 4$, $m = 3$, $u = 5$, $v = 2$, find w.

4 $M = 0.8$, $m = 0.6$, $u = 2$, $v = 5$, find w.

5 $M = 3$, $m = 2$, $u = 5$, $v = -2$, find w.

6 $M = 2$, $m = 3$, $u = -4$, $v = 7$, find w.

7 Car B, which is of mass 1200 kg, is initially stationary and is struck by car A, of mass 1500 kg, moving with speed 15 ms^{-1}. The cars become entangled and move together immediately after collision. Find their common speed after collision.

8 A railway truck of mass 12 tonnes moving with speed 2 ms^{-1} collides with a stationary truck, mass 16 tonnes. The trucks move together immediately on impact. Find their common speed.

9 A railway goods wagon A of total mass 20 tonnes is moving along the horizontal track in a railway yard at a speed of 1.5 kmh^{-1}. A second goods wagon B with a total mass of 25 tonnes and moving with speed 3 kmh^{-1} overtakes it and is coupled to it. Find the common velocity v of the two wagons as they move together after being coupled.

10 A 0.045 kg rifle bullet is fired horizontally with a velocity of 425 ms^{-1} into a 5 kg block of wood which can move freely in the horizontal direction. Determine the final velocity of the block.

11 A girl of mass 44 kg runs and, when her horizontal velocity is 5 ms^{-1}, jumps on a stationary 16 kg sledge. The girl and sledge travel a distance of 18 m horizontally on snow before coming to rest. What is the coefficient of sliding friction between the toboggan and the snow?

6.4 Elastic collisions

There are effectively two different situations, one where one body remains fixed (ball hitting a wall) and the other where both bodies can move. They will be examined in turn.

One body fixed

When a ball, for example, hits a wall there is no theoretical method for finding the speed just after leaving the wall. Experiments have however shown that if a ball, or particle, is moving with speed u perpendicular to the wall just before impact, then the speed with which it leaves the wall is eu, where e is a number depending on the elastic properties of both the wall and the ball and is known as the coefficient of restitution. This experimental law was first established by Newton.

The speed u with which the ball is approaching the wall is referred to by Newton as the speed of approach, and he referred to v, the speed of the ball as it leaves the wall, as the speed of separation. He expressed his experimental law in the form

$$\frac{\text{speed of separation}}{\text{speed of approach}} = e,$$

where e is the coefficient of restitution and satisfies the conditions $0 \leq e \leq 1$. The lower limit refers to a perfectly plastic, or inelastic collision, where the ball sticks to the wall. The upper limit corresponds to a perfectly elastic collision where the ball leaves the wall with speed u.

Example 6.6

A ball is dropped vertically downwards onto a smooth plane from a height of 1.4 m. Given that the coefficient of restitution between the ball and plane is 0.6, find

(i) the height to which the ball first bounces,

(ii) the time taken from dropping the ball to it reaching the top of its first bounce.

The formula $v^2 = u^2 + 2gh$, gives that the speed, in ms^{-1}, of the ball when it first reaches the plane is $\sqrt{2 \times 9.8 \times 1.4} = 5.24$.

The speed of rebound is, from Newton's law,

$$0.6 \times 5.24 = 3.14.$$

Applying the above formula again gives the height, in m, to which the ball rises as

$$\frac{3.14^2}{2 \times 9.8} = 0.50.$$

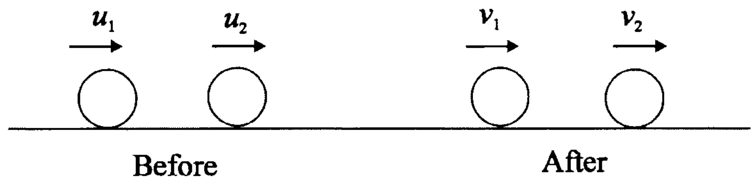
Applying the formula $v = u + gt$ gives the time, in seconds, of the downwards motion as $\frac{5.24}{9.8} = 0.53$. Applying the formula to the upwards motion gives the time, in seconds, to the top of the bounce as $\frac{3.14}{9.8} = 0.32$. The total time is therefore 0.85 s.

Both bodies free to move

Newton carried out experiments involving moving objects and found that his experimental law in the form

$$\frac{\text{speed of separation}}{\text{speed of approach}} = e, \quad \text{was still valid.}$$

Some care is needed in interpreting this when both bodies are moving.



The diagram shows two bodies moving along a line with velocity components u_1 and u_2. For $u_1 > u_2$ they will collide and the corresponding components after collision are v_1 and v_2. The speed of approach is the rate at which the distance between the two is decreasing before collision and this is $u_1 - u_2$. The speed of separation is the rate at which the distance between the two is increasing after collision and this is $v_2 - v_1$. Then Newton's law can be written as

$$v_2 - v_1 = e(u_1 - u_2) = -e(u_2 - u_1).$$

In solving practical problems it is important to pick a reference direction and calculate all the components in that direction. It is unwise to try and guess whether after collision a body is moving in a particular direction and show this direction in a diagram. If it turns out that a body is moving in the opposite direction to the reference one then this will show in calculations by the component being negative. It is easier to remember Newton's law in the form

$$v_2 - v_1 = -e(u_2 - u_1),$$

since always putting the minus sign outside avoids you having to remember to change the order of subtraction on the two sides of the equation.

In a collision problem the total momentum is conserved. This means that, if the masses of the two particles above are denoted by m_1 and m_2, then

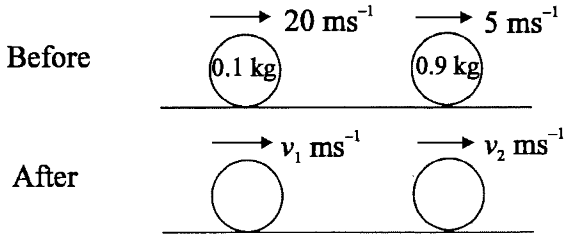
$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2.$$

All problems involving elastic collisions reduce to solving two equations of the above type.

Example 6.7

A small smooth sphere of mass 0.1 kg moving with speed of 20 ms^{-1} on a horizontal plane catches up and collides with a smooth sphere of the same radius but of mass 0.9 kg and moving with a speed of 5 ms^{-1}. The coefficient of restitution is $\frac{1}{3}$. Find the speeds of the spheres immediately after collision.

The most important step is to draw diagrams to show the motion before and after collision and to mark in a reference direction. There is no real point in trying to guess the directions of motion after collision and it is best to take all unknowns in the reference direction since it avoids problems with signs.



In the diagram the velocities of the spheres after collision are v_1 ms^{-1} and v_2 ms^{-1} to the right.

The component of the momentum, in Ns, to the right before collision is

$0.9 \times 5 + 0.1 \times 20 = 6.5$ and after collision it is $0.1v_1 + 0.9v_2$.

Conservation of momentum gives

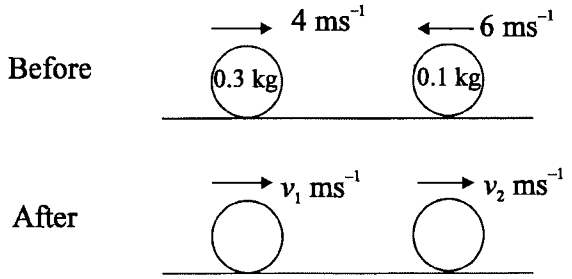
$$6.5 = 0.1v_1 + 0.9v_2.$$

Newton's law gives $$v_2 - v_1 = -\frac{1}{3}(5-20) = 5.$$

Solving these gives $$v_1 = 2, \quad v_2 = 7.$$

Example 6.8

Two small smooth spheres of equal radius but of mass 0.3 kg and 0.1 kg are moving directly towards each other with speeds of 4 ms^{-1} and 6 ms^{-1}, respectively, with the coefficient of restitution being 0.5. Find their speeds after collision.



The diagram shows the velocities before and after collision. In calculating the momentum and using Newton's law it is important to remember that the component of the initial velocity of the lighter sphere is – 6 ms^{-1} to the right. The component of the momentum, in Ns, to the right before collision is $0.3 \times 4 - 0.1 \times 6 = 0.6$ and after collision it is $0.3v_1 + 0.1v_2$.

Conservation of momentum gives

$$0.3v_1 + 0.1v_2 = 0.6.$$

Newton's law gives

$$v_2 - v_1 = -0.5(-6-4) = 5.$$

Solving these equations gives $v_1 = 0.25$, $v_2 = 5.25$.

Exercises 6.3

Questions 1 to 4 refer to a small ball dropped from rest at a height h onto a smooth plane, the coefficient of restitution being e, and h_1 being the height reached after the first bounce.

1 $h = 5$ m, $e = 0.4$, find h_1.

2 $h = 8$ m, $h_1 = 5$ m, find e.

3 $h = 1$ m, $e = \frac{1}{4}$, find the total distance travelled by the ball before it comes to rest.

4 $h = 2.4$ m, $e = 0.4$; find, given that the ball is of mass 0.2 kg, the magnitude of the impulse on the ball at the first bounce.

5 A tennis ball is projected vertically downwards from a height of 1.6 m onto a tennis court. Given that the coefficient of restitution is 0.8 find the speed of projection in order that the ball just returns to the point of projection after bouncing once.

In questions 6 to 8, a smooth sphere of mass m_1 kg moving with speed u_1 ms^{-1} catches up and collides directly with a smooth sphere of mass m_2 kg moving with speed u_2 ms^{-1} in the same direction. The velocity components after impact and measured in the original direction of motion are denoted by v_1 ms^{-1} and v_2 ms^{-1}, respectively.

6 $m_1 = 3$, $m_2 = 1$, $u_1 = 6$, $u_2 = 1$, $e = 0.4$, find v_1 and v_2.

7 $m_1 = 2$, $m_2 = 3$, $u_1 = 6$, $u_2 = 2$, $v_1 = 3$, find e and v_2.

8 $m_2 = 10$, $u_1 = 9$, $u_2 = 2$, $v_1 = 2$, $v_2 = 5$, find e and m_1.

In questions 9 to 11, a smooth sphere of mass m_1 kg moving with speed u_1 ms^{-1} collides directly with a smooth sphere of mass m_2 kg moving with speed u_2 ms^{-1} in the opposite direction. The velocity components after impact and measured in the direction of motion of the first sphere are denoted by v_1 ms^{-1} and v_2 ms^{-1}, respectively.

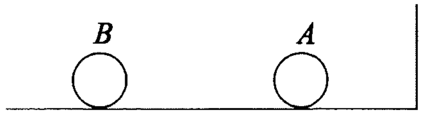
9 $m_1 = 4$, $m_2 = 1$, $u_1 = 3$, $u_2 = 1$, $e = 0.5$, find v_1 and v_2.
10 $m_1 = 4$, $u_1 = 3$, $u_2 = 5$, $v_1 = 2$, $e = 0.2$, find m_2 and v_2.
11 $m_2 = 12$, $u_1 = 10$, $u_2 = 2$, $v_1 = 2$, $v_2 = 4$, find e and m_1.
12 A smooth sphere of mass 4 kg collides directly with a smooth sphere of mass 8 kg which is at rest. Find the condition to be satisfied by e in order that the spheres move in opposite directions after collision.
13 Identical cars A, B, C are in a straight line, very slightly apart, with their brakes off. Car A is pushed towards the others so that it hits car B with speed 1 ms^{-1}. Given that the coefficient of restitution is 0.75, find the speeds of the cars after all collisions have finished.

Miscellaneous Exercises 6

1 The police are investigating an accident where a moving car of mass 1000 kg collided directly with a parked one of mass 1200 kg. Immediately after the collision, the cars became enmeshed and slid forward together a distance of 10 m. Assuming that the coefficient of friction is 0.5, that the road is level and that the only horizontal force acting on the enmeshed cars is friction, find the common speed of the cars as they start moving. Hence find the speed of the moving car at the instant of collision.
2 A lorry of mass 4 tonnes is towing a car of mass 1200 kg. The lorry sets off with the tow rope slack but when its speed is 1.5 ms^{-1} the tow rope tightens. Find
(i) the speed of the car immediately after it starts moving,
(ii) the impulsive tension in the tow rope.
3 A car of mass 1200 kg moving with speed 30 ms^{-1} crashes into the back of a car of mass 1000 kg moving in the same direction with speed 15 ms^{-1}. After collision, the cars move together. Find their common speed immediately after the collision and the impulse on the lighter car.
4 Two particles P and Q of masses $4m$ and $5m$, respectively, are attached one to each end of a light inextensible string which passes over a small smooth pulley. The particles move in a vertical plane with both hanging parts vertical and they are released from rest. Find in terms of m and/or g, as appropriate, the magnitudes of the accelerations of the particles and the tension in the string.
When the particle P is moving upwards with speed V it picks up from a point A an additional particle of mass $2m$ so as to form a composite particle R of mass $6m$. Find the initial speed of R.

5 A small smooth sphere is dropped from a point at a height of 0.6 m above a smooth horizontal floor. The sphere falls vertically, strikes the floor and bounces to a height of 0.15 m above the floor. Find
(a) the speed of the sphere when it hits the floor,
(b) the coefficient of restitution between the sphere and the floor.
6 A small smooth sphere P of mass $6m$ moving on a smooth plane in a straight line with constant speed $8u$ collides directly with a small smooth sphere Q of the same radius but of mass $4m$ and moving in the same direction with speed $6u$. The direction of motion of sphere Q is unchanged by the collision and immediately after the collision it moves with speed $8u$. Find
(i) the speed of P immediately after collision,
(ii) the coefficient of restitution.
7 Two small smooth spheres A and B of equal mass moving in the same straight line (and in the same direction) and with speeds $2u$ and u, respectively, collide directly. After the collision, their directions of motion remain unchanged but their speeds are v and $1.5v$ respectively. Show that the coefficient of restitution is 0.6.
8 Sphere P moves on a horizontal floor and collides directly with an identical sphere Q which is at rest at a distance of 2 m from a smooth vertical wall, Q being nearer to the wall than P. The motions before and after all possible collisions are perpendicular to the wall and the coefficient of restitution for all collisions is 0.6.
(i) Show that when Q collides with the wall for the first time, P is at a distance of 1.5 m fom the wall.
(ii) Find the distance of the spheres from the wall when they collide for the second time.
9 Two small smooth spheres P and Q of equal radius but of mass $2m$ and $4m$, respectively, are moving directly towards each other on a smooth horizontal table with speeds $2u$ and $3u$, respectively. The collision is such that Q receives an impulse of magnitude $8amu$, where a is a constant. Find
(a) the speeds of the spheres immediately after collision,
(b) the coefficient of restitution,
(c) the range of possible values of a.
10 A small smooth sphere A of mass m and moving with speed $5u$ catches up and collides directly with a second sphere B of mass $4m$ and moving with speed u. After collision the direction of motion of A is reversed and its speed is u. Find
(a) the speed of B after collision,
(b) the coefficient of restitution.

11



The diagram shows two small smooth spheres A and B of equal radii at rest on a horizontal table, with sphere A between a smooth vertical wall and sphere B. The line joining the centres of the spheres is perpendicular to the wall. Sphere A is of mass m and sphere B is of mass qm. Sphere A is projected so as to collide directly with sphere B. Given that the coefficient of restitution between the spheres is $\frac{2}{5}$ and $q > \frac{5}{2}$, show that the direction of motion of sphere A is reversed by the impact.
Given, additionally, that the coefficient of restitution between sphere A and the wall is $\frac{1}{5}$, find the condition to be satisfied by q so that the spheres will collide again.
(You may assume that, between collisions, the spheres move with constant speed.)

12 Two small smooth spheres P and Q of equal radii but of masses m and $3m$, respectively, are moving towards each other on a smooth horizontal table. Before collision, the speeds of P and Q are $3u$ and $6u$, respectively, and after collision the direction of motion of P is reversed and it moves with speed $5u$. Find the impulse on Q, the speed of Q after collision and the coefficient of restitution between P and Q.

13 Two small smooth spheres A and B of mass $2m$ and $5m$, respectively, and moving along Ox collide. The velocity of A immediately before collision is $5u$ in the positive x-direction and immediately after collision the velocities of A and B in the positive x-direction are $2u$ and $4u$ respectively. Determine
(i) the velocity of B immediately before collision,
(ii) the magnitude of the impulse on B,
(ii) the value of the coefficient of restitution.

14 Two small smooth spheres A and B of equal radii, but of masses m and km, respectively, are at rest on a smooth horizontal floor. The sphere B lies between sphere A and a smooth vertical wall and is at a perpendicular distance of 3 m from the wall. The line AB is perpendicular to the wall. The coefficient of restitution for collisions between the spheres and between sphere B and the wall is 0.2. Sphere A is then projected directly towards sphere B with speed u ms^{-1}. Find the velocities of the spheres immediately after impact.
(i) Given that $k = 3$, find
 (a) the distance between A and B when B hits the wall,
 (b) the distance from the wall to the point at which the spheres next collide.

(ii) Given that $k \neq 3$, find the least value of k so that the spheres do not collide after B hits the wall.

15 Two small smooth balls A and B of masses 0.2 kg and 0.1 kg, respectively, are moving in a straight vertical line through a fixed point O. The balls collide at a point P above O. Immediately before collision, ball A is moving downwards and ball B is moving upwards. The coefficient of restitution between the balls is $\frac{3}{5}$ and immediately after collision ball A moves upwards with a speed of 1 ms^{-1} and ball B moves downwards with a speed of 17 ms^{-1}. Find
(i) the speeds of both balls immediately before collision,
(ii) the impulse imparted to A by the collision.
Explain why your results are consistent with both balls being projected from a point below P with the same initial speed, with A being projected before B.

Chapter 7

Centre of Mass

After working through this chapter you should be able to

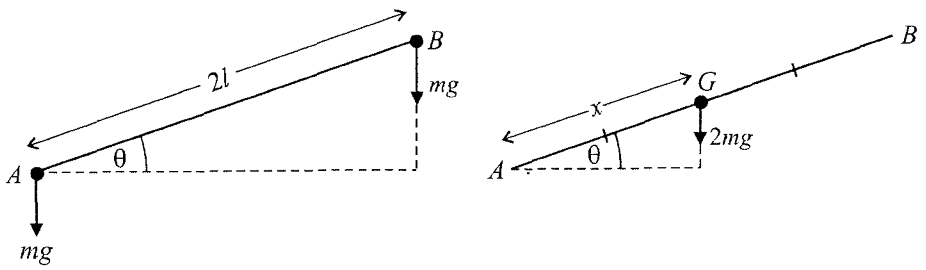
- find the centre of mass of a coplanar system of particles,
- find the centre of mass of a uniform plane lamina,
- solve problems involving equilibrium of a suspended plane lamina.

7.1 Coplanar system of particles

The centre of mass of a number of particles lying on a plane is the point through which the resultant weight of the system acts. Equivalently, the total moment of the weights about the centre of mass is zero.

Example 7.1

Two particles of equal mass m are at points A and B.

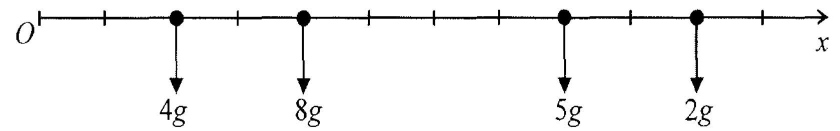


This system is equivalent to a mass of 2m at G, the centre of mass. Suppose distance AB is $2l$. Total moments about A of the particles is $mg \times 0 + mg \times 2l\cos\theta = 2mgl\cos\theta$. Moments about A of the equivalent mass of 2m at G is $2mg \times x\cos\theta = 2mgx\cos\theta$. Equating the moments gives $x = l$ so that G is the mid point between A and B. Equivalently, the total moment of the system about G is $mg \times l\cos\theta - mg \times l\cos\theta = 0$.

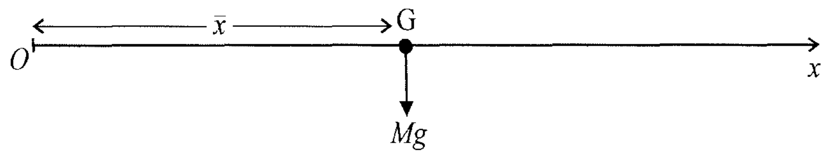
Note that g may be left out of the equations without affecting the answer.

Example 7.2

Find the centre of mass of particles with masses 4kg, 8kg, 5kg and 2kg which lie on the x-axis at points (2,0), (4,0), (8,0) and (10,0).



Suppose the resultant force Mg acts through G on the x-axis at a distance $\bar{x}$ from origin O.



Resolving vertically gives

$$\begin{aligned} Mg &= 4g + 8g + 5g + 2g \\ M &= 19 \end{aligned}$$

Equating the moments of the two equivalent systems about O gives

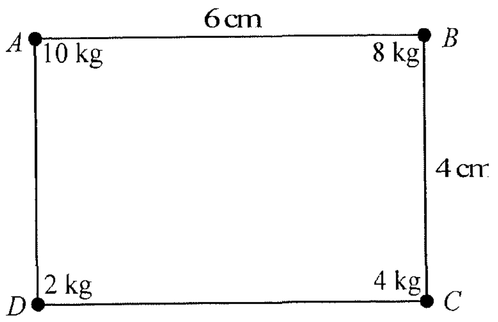
$$\begin{aligned} Mg \times \bar{x} &= (4g \times 2) + (8g \times 4) + (5g \times 8) + (2g \times 10) \\ 19\bar{x} &= 100 \\ \bar{x} &= 5.26 \end{aligned}$$

The centre of mass is at a point 5.26 units from the origin. Note again that the acceleration due to gravity g cancels out in the equation, and may be left out of the problem without affecting the answer.

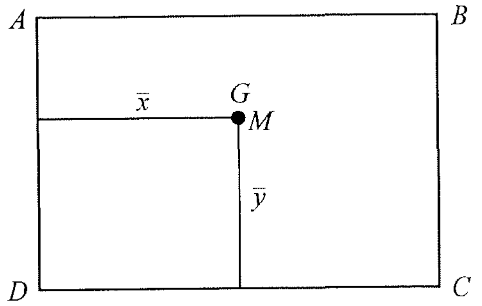
When the masses are not all on a straight line, but scattered on a plane, two equations of moments are necessary. Usually, these are taken at right angles to each other.

Example 7.3

The rectangle *ABCD* has sides *AB*=6cm and *BC*=4cm. Particles of masses 10kg, 8kg, 4kg and 2kg are placed at *A*, *B*, *C* and *D* respectively. Find the position of the centre of mass of the system.



Suppose the centre of mass is at *G*, where $\bar{x}$ and $\bar{y}$ are distances of *G* from *AD* and *DC* respectively.



The resultant weight of the system, Mg, acts through *G* perpendicular to the plane *ABCD*.
Resolving perpendicular to plane *ABCD*

$$\begin{aligned} Mg &= 10g + 8g + 4g + 2g \\ M &= 24 \end{aligned}$$

Equating moments of the two systems about *AD*

$$\begin{aligned} Mg \times \bar{x} &= (10g \times 0) + (8g \times 6) + (4g \times 6) + (2g \times 0) \\ 24\,\bar{x} &= 72 \\ \bar{x} &= 3\text{cm} \end{aligned}$$

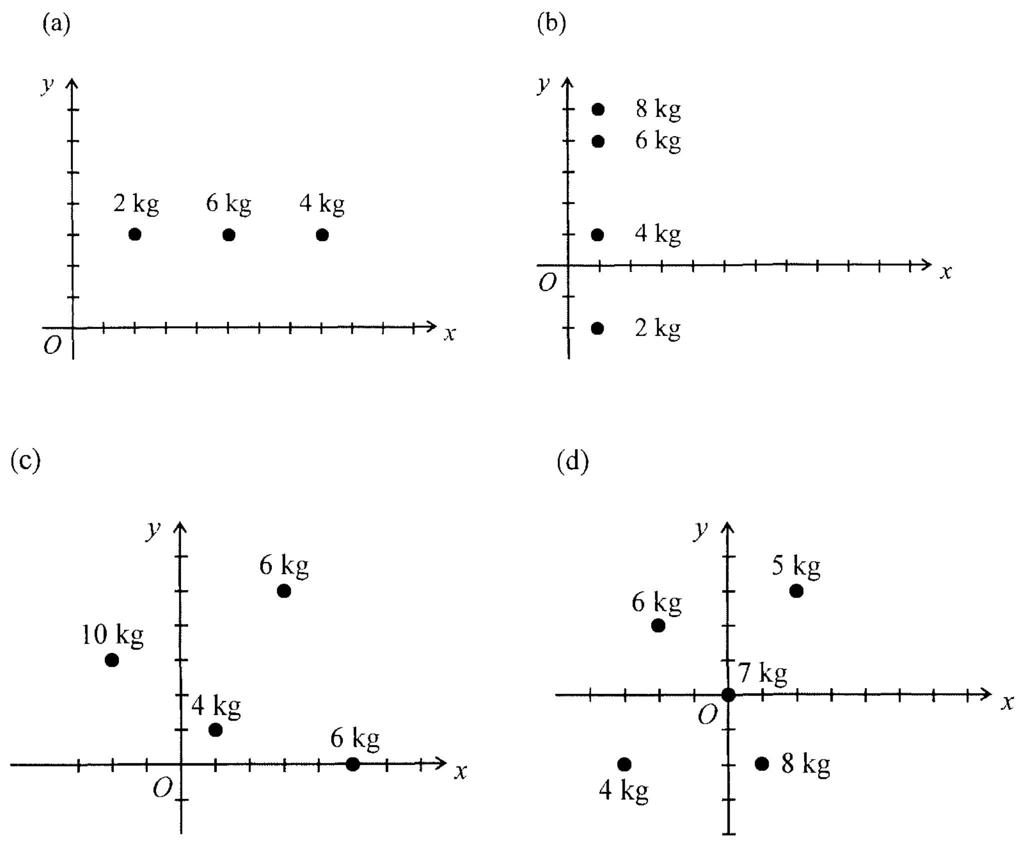
Equating moment of the two systems about *DC*

$$\begin{aligned} Mg \times \bar{y} &= (10g \times 4) + (8g \times 4) + (4g \times 0) + (2g \times 0) \\ 24\,\bar{y} &= 72 \\ \bar{y} &= 3\text{cm} \end{aligned}$$

Therefore the centre of mass is at a point 3cm from *AD* and 3cm from *DC*.
Note that the acceleration due to gravity g cancels out in all the equations. M is simply the sum of all the masses of the particles.

Exercises 7.1

1 Find the co-ordinates of the centre of mass of the following system of particles.



In each of the exercises 2 – 4 below, find the position of the centre of mass of the system, giving your answers from the points of reference suggested.

2 Particles of masses 4, 7, 10, 13kg at *A*, *B*, *C*, *D* where *ABCD* is a straight line with *AB*=*BC*=*CD*=5cm. (Distance from *A*.)

3 Particles of masses 3, 5, 4kg at *A*, *B*, *C* where *ABC* is a triangle in which *AC*=5m, *AB*=3m and *BC*=4m. (Distances from *AB* and *BC*.)

4 Particles of masses *m*, 2*m*, 3*m*, 4*m* and 6*m at A*, *B*, *C*, *D* and *O* respectively, where *ABCD* is a rectangle with *A*B=6m, *BC*=8m and *AC* and *BD* intersect at *O*. (Distances from AB and *DC*.)

5 Particles of masses 7kg, 3kg, 5kg, 6kg, m kg and M kg are situated at the vertices of a regular hexagon *ABCDEF* respectively. The centre of mass of the system is at *G* which is on *CF* such that *CG* : *GF* = 3 : 1. Find m and M.

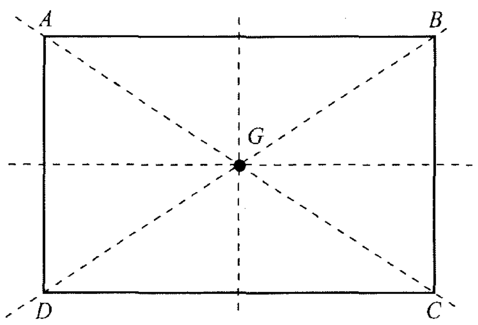
7.2 Lamina

Up to now, we have dealt only with particles. We now consider laminas which are 2-dimensional shapes. The thickness is assumed to be negligible compared to the other dimensions. It can be shown that there is a point in every lamina, namely its centre of mass, at which its mass may be supposed to be concentrated. It is the point through which the weight of the lamina acts.

Uniform Lamina

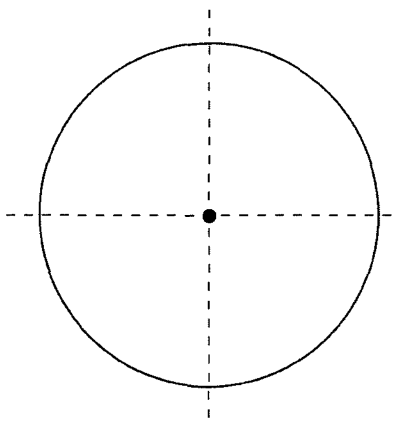
A uniform lamina is one in which the mass is proportional to its area. We shall only consider simple shapes such as the rectangle, the circle, the triangle and composite shapes made up of a combination of these, either added on or cut out of another such shape. In practice, most bodies are not perfectly uniform. Uniformity is a modelling assumption.

Rectangle



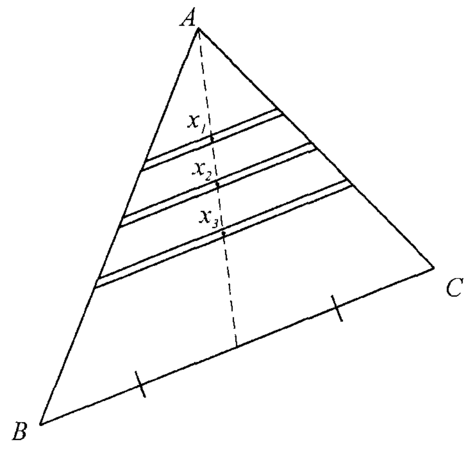
The centre of mass of a rectangular uniform lamina is at the geometrical centre of the rectangle, i.e. the intersection of its diagonals, or the intersection of its two lines of symmetry.

Circle

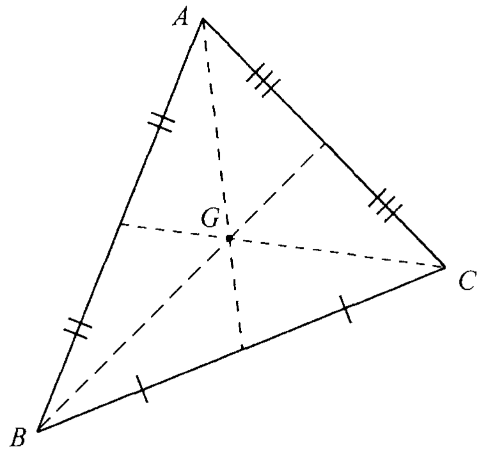


The centre of mass of a circular lamina is at the centre of the circle.

Triangle



Suppose the triangular uniform lamina ABC is divided into large numbers of thin parallel strips by lines parallel to BC. The centre of mass of each strip lies on the centre x_1, x_2, x_3 etc of each strip. These points x_1, x_2 etc all lie on the median joining A to the midpoint BC. Therefore, the centre of mass G lies on the median.

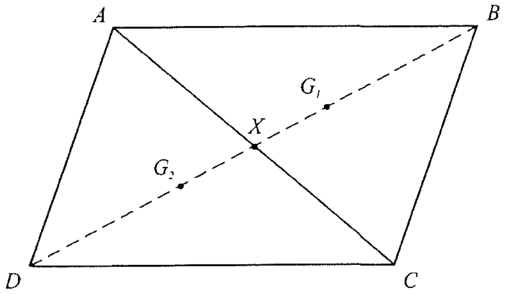


Similarly, the centre of mass G also lies on the other two medians.
Therefore, the centre of mass of a triangular uniform lamina lies on the intersection of the three medians of the triangular. This is $\frac{2}{3}$ of the length of the median measured from the vertex.

Usually, problems involve composite shapes made up of these basic shapes. For example, a triangle may be added onto a rectangle to make a shape like a Christmas tree. A circular hole may be removed from another shape. There may be particles placed on points on the lamina, or a shape may be made up of uniform wire. All of these problems resolve into problems of centre of mass of a coplanar system of particles; each simple shape giving rise to a mass at its centre of mass.

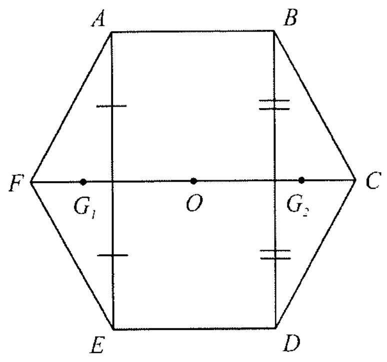
It is worth considering the centre of mass of some common regular shapes. Some examples are given below.

Parallelogram



The diagonals AB and CD bisect each other. The parallelogram is made up of two triangles ABC and ADC. The centre of mass G_1 of the triangle ABC lies on BD, as does the centre of mass G_2 of the triangle ADC. Also XG_1 equals XG_2, so that X is also the centre of gravity of $ABCD$.

Hexagon

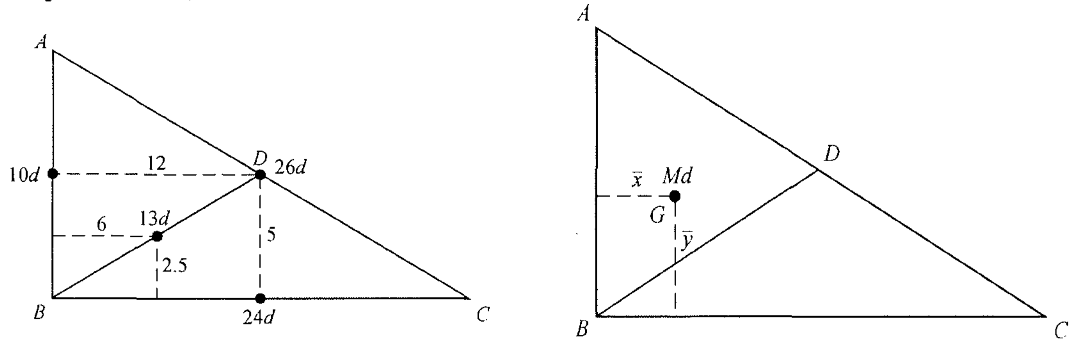


The regular hexagon $ABCDEF$ can be divided into triangles AEF, BCD and rectangle $ABDE$. O is the centre of mass of rectangle $ABDE$. The centre of mass G_1 of triangle AEF lies on FC as does the centre of mass G_2 of triangle BCD. By symmetry $OG_1 = OG_2$ and so O is also the centre of mass of the hexagon.

In general, the geometrical centre of a regularly shaped uniform lamina is also its centre of mass.

Example 7.4

Find the centre of mass of a frame made of uniform wire in the shape of a right-angled triangle ABC with sides AB=10cm, BC=24cm and AC=26cm with a strut joining D, the mid-point of AC, to B.



Although the problem involves uniform wire, the masses behave as if they are particles located at the centre of mass of each piece of wire. Let the density of the wire be d kg per cm. The problem reduces to one of particles at a plane. AB, BC, AC and BD have masses $10d$, $24d$, $26d$ and $13d$ kg located at its midpoint as shown in the diagram. The total mass $Md = 10d + 24d + 26d + 13d = 73d$ kg at G a distance $\bar{x}$ from AB and $\bar{y}$ from BC.

Equating moments of the two systems about AB.

$$M\bar{x} = (10 \times 0) + (13 \times 6) + (26 \times 12) + (24 \times 12)$$
$$73\bar{x} = 678$$
$$\bar{x} = 9.3\text{cm}$$

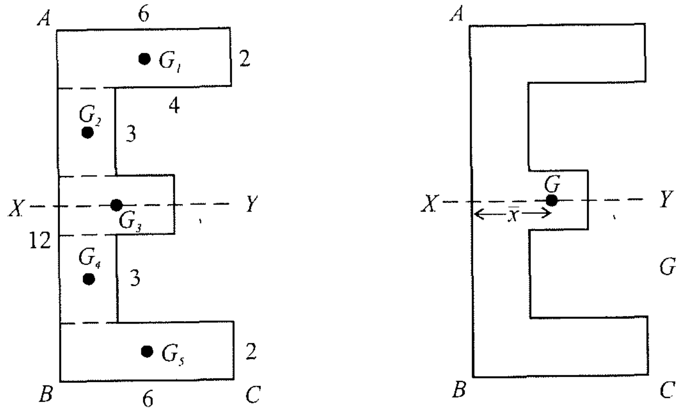
Equating moments of the two systems about BC.

$$M\bar{y} = (24 \times 0) + (13 \times 2.5) + (10 \times 5) + (26 \times 5)$$
$$73\bar{y} = 212.5$$
$$\bar{y} = 2.9\text{cm}$$

The centre of mass is 9.3cm from AB and 2.9cm from BC.

Example 7.5

Find the centre of mass of the piece of cardboard cut into the shape shown in the diagram. Lengths are in cm.



The shape may be divided into 5 rectangles each with centre of mass at its geometric centre shown as G_1, G_2, G_3, G_4, and G_5 in the above diagram. It has a line of symmetry XY, a distance of 6cm from BC and the centre of mass G will lie on it. Let the density of the cardboard be d kg per cm^2.

Total mass $\quad Md \quad = \quad (6 \times 2)d + (3 \times 2)d + (4 \times 2)d + (3 \times 2)d + (6 \times 2)d$

$\qquad\qquad\quad M \quad = \quad 44$

Suppose centre of mass G is a distance $\bar{x}$ cm from AB.

Equating moments of the two systems about AB.

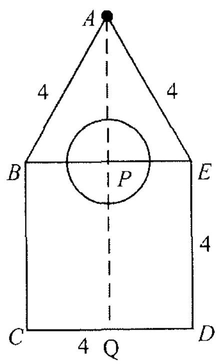
$$M \times \bar{x} = (12 \times 3) + (6 \times 1) + (8 \times 2) + (6 \times 1) + (12 \text{ x } 3)$$

$$44\bar{x} = 100$$

$$\bar{x} = 2.27$$

The centre of mass is a distance 2.27cm from AB and 6cm from BC.

Example 7.6



The diagram shows a uniform lamina made of an equilateral triangle ABE joined to a square $BCDE$. All the sides are of length 4cm. P, Q are the midpoints of BE and CD respectively. A circle centre P and radius 1cm is removed from the lamina. A particle of mass 0.8kg is attached to the point A. The material has density 0.2kg per cm^2.

Find the centre of mass of the lamina.

There is a line of symmetry AQ and the centre of mass G will lie on it.

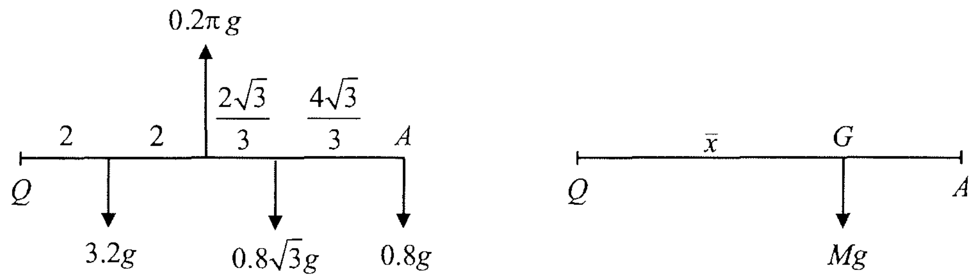
Triangle ABE has area $4\sqrt{3}$ cm^2, mass $0.8\sqrt{3}$ kg acting at $\frac{2}{3} \times 4 \sin 60^\circ = \dfrac{4\sqrt{3}}{3}$ cm from A.

Square $BCDE$ has area 16cm^2, mass 3.2kg acting at 2cm from Q.

Particle A has mass 0.8kg acting at A.

Circle centre P has area π cm^2, mass 0.2 π kg acting at P.

Suppose the centre of mass G is at a distance $\bar{x}$ from Q.



Total mass $\quad$ M $\quad = \quad 3.2 - 0.2\pi + 0.8\sqrt{3} + 0.8$

$\qquad\qquad\qquad\quad = \quad 4 - 0.2\pi + 0.8\sqrt{3}$

Equating moments of the two system about Q.

$$\text{M} \times \bar{x} = (3.2 \times 2) - (0.2\pi \times 4) + (0.8\sqrt{3} \times (4 + \frac{2\sqrt{3}}{3})) + (0.8 \times (4 + 2\sqrt{3}))$$

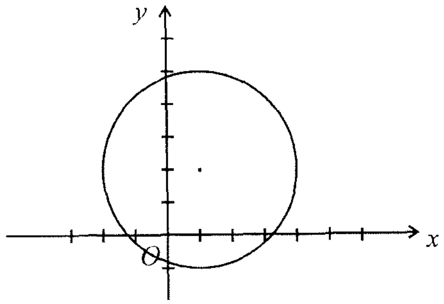
Substituting the value of M into this equation gives $\bar{x} = 4.23$.

Centre of gravity lies along AQ at 4.23cm from Q.

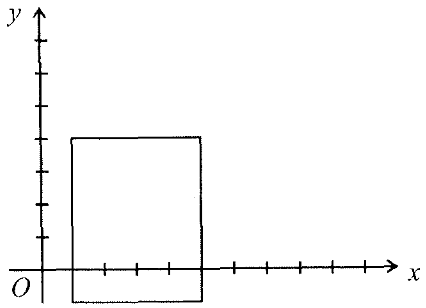
Exercises 7.2

1 Write down the co-ordinates of the centre of mass of the following laminae.

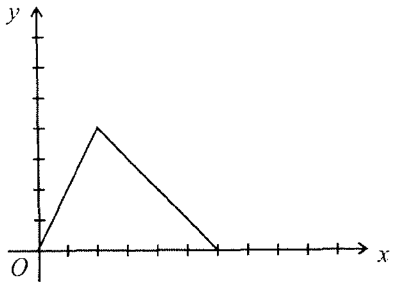
(a)



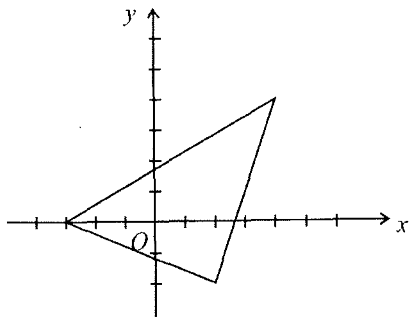
(b)



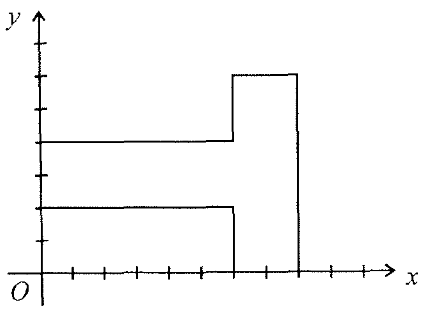
(c)



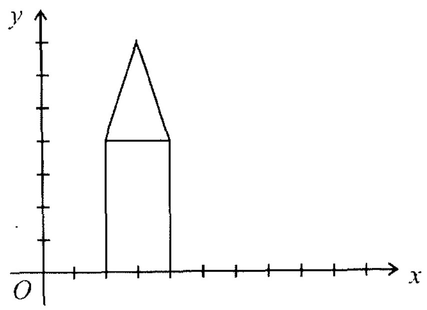
(d)



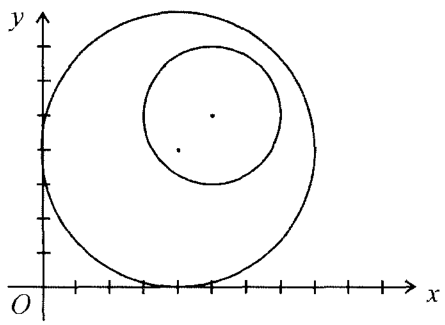
(e)



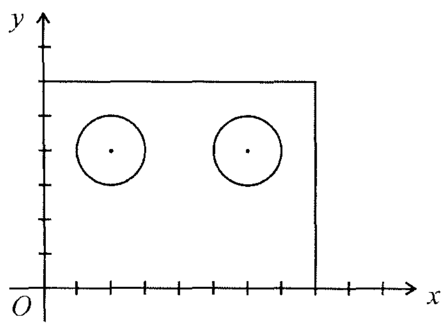
(f)



(g)



(h)



2 *ABCD* is a square sheet of paper centre *O*, of side 24cm. The corners *A* and *B* are folded over so that they coincide at *O*. Find the position of the centre of mass of the folded sheet.

3 *AB* and *CD* are parallel sides of a trapezium of lengths 4cm and 6cm and a distance 10cm apart. Find the distance of the centre of mass from *CD*.

7.3 Equilibrium of a suspended lamina

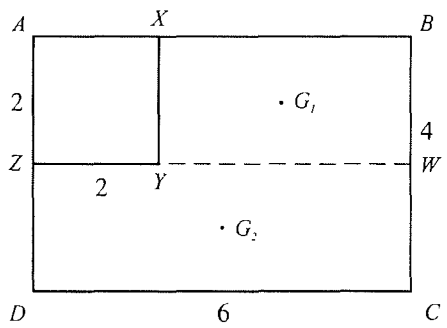
If a lamina is freely suspended, it rests in equilibrium under the action of its resultant weight acting vertically downwards through its centre of mass, and the tension in the string. The two forces are equal and opposite, and the line joining the point of suspension and the centre of mass must be vertical.

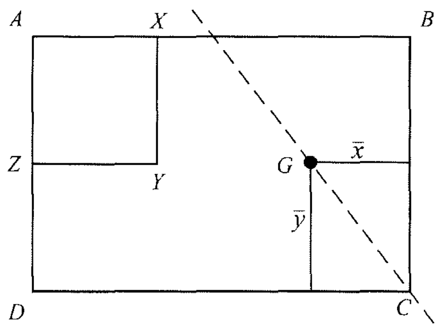
Example 7.7

ABCD is a uniform rectangular lamina such that AB=6cm and BC=4cm. A square of edge 2cm with one corner at *A* is cut away. The lamina is suspended in equilibrium by means of a string attached at *C*.

Find the angle *CB* makes with the vertical.

First find the centre of mass of the lamina.





Mass of rectangle *XYWB* is 8 units acting at G_1 2cm from *BC* and 3cm from *DC*

Mass of rectangle *ZDCW* is 12 units acting at G_2 3cm from *BC* and 1cm from *DC*

Total mass of Lamina M = 20 units.

Equating moments of the two systems about *BC*.

$$M \times \bar{x} = (8 \times 2) + (12 \times 3)$$

Substituting for M gives

$$\bar{x} = 2.6$$

Equating moments of the two systems about *DC*.

$$M \times \bar{y} = (8 \times 3) + (12 \times 1)$$

Substituting for M gives

$$\bar{y} = 1.8$$

The centre of mass *G* of the lamina is 2.6cm from *BC* and 1.8cm from *DC*.

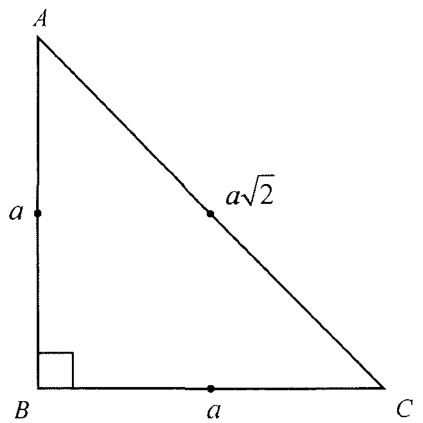
When the lamina is suspended from *C*, *CG* is vertical. So the angle *CB* makes with the vertical is $\angle BCG$.

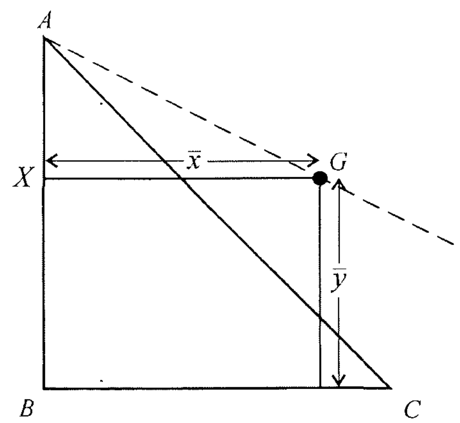
$$\tan \angle BCG = \frac{\bar{x}}{\bar{y}} = \frac{2.6}{1.8}$$

This gives $\angle BCG = 55.3°$, so *CB* makes an angle of 55.3° with the vertical when the lamina is suspended freely from *C*.

Example 7.8

A right-angled triangle *ABC* is made with uniform rods, *AB* = a m, *BC* = a m and $\angle ABC$=90°. Find the distance from *BA* and *BC* of the centre of mass of the three rods. If the triangle is suspended in equilibrium from the point *A*, show that *AB* will make an angle of 28.7° with the vertical.





Mass of *AB* = a units² acting at mid-point of *AB*.
Mass of *BC* = a units² acting at mid-point of *BC*.
Mass of *AC* = $a\sqrt{2}$ units² acting at mid-point of *AC*
Total mass = a $(2 + \sqrt{2})$ units² acting at *G* distances $\bar{x}, \bar{y}$ from *AB, BC*

Equating moments of the two systems about *AB*.

$$a(2 + \sqrt{2})\,\bar{x} = (a \times 0) + (a \times \frac{a}{2}) + (a\sqrt{2} \times \frac{a}{2})$$

$$\bar{x} = \frac{a(1+\sqrt{2})}{2(2+\sqrt{2})}$$

By symmetry $$\bar{y} = \frac{a(1+\sqrt{2})}{2(2+\sqrt{2})}.$$

When the system is suspended from *A*, *AG* is vertical and the angle *AB* makes with the vertical is $\angle BAG$.

$$\tan \angle BAG = \frac{XG}{XA} = \frac{\bar{x}}{a - \bar{y}}$$

$$= \frac{\frac{(1+\sqrt{2})}{2(2+\sqrt{2})}}{\frac{1-(1+\sqrt{2})}{2(2+\sqrt{2})}}$$

$$= \frac{1+\sqrt{2}}{3+\sqrt{2}}.$$

This gives the answer $\angle BAG = 28.7°$, so that *AB* makes an angle of 28.7° with the vertical when the lamina is suspended freely from *A*.

Exercises 7.3

1 *ABCD* is a square lamina of negligible mass and *X*, *Y* are the mid-points of *AB* and *BC* respectively. Particles of masses 1, 2, 3, 4 and 5 kg are attached at *X*, *B*, *Y*, *C* and *D* respectively. The system is suspended in equilibrium from *A*. Find, to the nearest degree, the angle between *AD* and the vertical.

2 A lamina *ABCD* is in the shape of a rhombus of sides 5cm and *AC*=8cm. Triangle *ABC* is a uniform lamina with mass M and triangle *ACD* is a uniform lamina with mass 3M. The system is suspended in equilibrium from *A*. Calculate the angle *AC* makes with the vertical.

A mass kM is added to the point *B* so that when suspended freely from *A*, *AC* is vertical. Find the value of k.

3 Two uniform rods *AB* and *BC* of equal length 2a and of equal mass M are rigidly connected at *B* such that $\angle ABC = 90°$. The rods are suspended in equilibrium from *A*. Find the tangent of the angle *AB* makes with the vertical.

If a mass kM is added to the end *C*, show that the tangent of the angle *AB* makes with the vertical is $\dfrac{(1+2k)}{(3+2k)}$.

Calculate k if this tangent is $\frac{3}{4}$.

4 A hexagon *ABCDEF* is made of uniform rods *AB*=*AF*=*DC*=*DE*=5cm, *BC*=4cm and *EF*=2cm. The distance *AD* is 10cm, $\angle ABC = \angle BCD$ and $\angle AFE = \angle FED$. Find the distance of the centre of mass of the system from *BC*.

The system is suspended in equilibrium from the point *B*. Calculate the angle *BC* makes with the vertical.

5 A uniform lamina consists of a circular disc centre *O* and radius *OA*=2a. A circular portion with *OA* as diameter is removed. The resulting lamina has mass M kg. Show that the centre of mass of the lamina is at a distance $\dfrac{a}{3}$ from *O*.

BC is the diameter perpendicular to *OA*. A particle of mass $\dfrac{M}{3}$kg is attached to the lamina at the point *C*. The system is freely suspended from *B* and is in equilibrium. Find the tangent of the angle which *OB* makes with the vertical.

6 A uniform rectangular metal sheet *ABCD* has *AB*=8a and *BC*=4a. *X* is the mid-point of *AD* and *Y* is the mid-point of *BC*. *P* and *Q* are two points that lie on *XY* such that *PQ*=2a. A circle with diameter *PQ* is removed. The remainder of the sheet is suspended from *B* and it is found that at equilibrium, *BP* is vertical. Show that the centre of mass of the remaining lamina is a distance $a(4-\dfrac{\pi}{32})$ from *AD*.

Answers to Exercises

Exercises 2.1

1.	2.85N, 10.6N
2.	-4N, -6.93N
3.	-4.79N, 13.2N
4.	5.13N, -14.1N
5(a).	-5N, -8.66N
(b).	-4.60N, -3.86N
6(a).	4N, -6.93
(b).	1.37N, -3.76N

Exercises 2.2

1.	7N, 2N
2.	14N, 6.93N
3.	1.41N, -0.101N
4.	1.73N, -2N
5.	-8.57N, -4.55N

Exercises 2.3

1.	12N
2.	α=0, P=6N
3.	P=11.3N, Q=4.10N
4.	P=17.1N, Q=12.8N
5.	5.96N, Q=4.39
6.	26N, 67°
7.	14.7N, 4°
8(i).	R=5N, $\tan\theta = \frac{3}{4}$
(ii).	R=7.28N, $\tan\theta = \frac{2}{7}$

Exercises 2.4

1(a).	3.92N, 3.92N
(b).	1.5kg, 14.7N
2(a).	29.4N
(b).	2kg
3.	0.4m
4.	1000N
5.	1.4m
6.	3.29N
7.	P=1.96N, T=3.39N
8.	2.78N, 3.08N
9.	$\dfrac{mg}{(1+\cos\alpha)}\sin\alpha$
10.	2.08m
11.	2.09m
12.	588N, 509N, horizontally, 882N, vertically
13.	261N
14.	3.53kg, 20N
15.	0.2
16.	9.8N, 9.8N and 13.7N, 0.11

Exercises 2.5

1(a).	30.6N
(b).	78.3N
2.	0.340
3(i).	7.84N (ii). 9.26N
4(i).	17.8N (ii). 27.2N
5(i).	102N (ii). 11.5N
6.	0.255

Exercises 2.6

1(a). 7.62N, 23.2°

(b). 8.94N, -63.4°

(c). 11.4N, 105°

(d). 13.9N, 249°

(e). 8.06N, 150.3°

(f). 4.24N, 225°

2(a). 3.74N, -77.8°

(b). 5.95N, 141°

(c). 5.92N, 72.5°

3(a). 3.74N at 102.2°

(b). 5.95N at –39°

(c). 5.92N at 252.5°

Miscellaneous Exercises 2

1(a). 20N, 21N

(b). 29N

(c). 46.4°

(d). 29N at 226.4° to x direction

2. 1, 7

3. 42N at 261.8° to AB

4. 22, 60°

5. light, 0.48m

6. 78N, 325N

7(i). $\sqrt{3}P$ (ii). 150° (iii). 120°, 60°

9. 16.7°

10. 4022N, 1105N

11. 31°

12. 0.24

13. 0.58

14. 11.4N, 13.9N

15(a). 80N

(b). 69.28N

16. 0.5

17. $\frac{1}{4}$, $\frac{3}{\sqrt{15}}$

18(i). 1975N (ii). 3890N (iii). 3712N

20(i). 1.96N (ii). 41.8°

21(i). 4.62N (ii). –0.63N

22. 1.73kN

Exercises 3.1

1. -6.4Nm, -2.4Nm

2. 3.8Nm, -7.6Nm

3. 1.6Nm, 9.6Nm

4. -3.3Nm, -3.9Nm

5. 5.2Nm

6. 26Nm

7. -3Nm

8. 18Nm

9. 11Nm

10. -1Nm

11. $2\theta\,\ell\cos\theta - W\ell\cos\theta$, $\theta\,\ell\cos\theta - P\ell\sin\theta$

12. $2\theta\,\ell\cos\theta - W\ell\cos\theta - F2\ell\sin\theta$,
$\theta\,\ell\cos\theta - P\ell\sin\theta - F\,\ell\sin\theta$

13. $4S\ell\cos\theta - 10W\ell\cos\theta$,
$6W\ell\cos\theta - 4R\ell\cos\theta$

14. $2Fa\sin\theta - Wa\sin 2\theta$, $2Ra - Wa\sin 2\theta$

Exercises 3.2

1. 10N, 6N

2. 13N, 4N

3. 1N, 15N

4. 6N, 3N

5. 4kg

6. $2\frac{2}{3}$m from B

7. $1\frac{2}{3}$m

8. 21.82N, 38.18N, 40N

9. m tan 15°

Exercises 3.3

1. 12N at 3.5m to right of O

2. 16N downwards at 3.375m to left of O.

3. 26N upwards at 8m to right of O.

4. 12N downwards at 4.5m right of O.

5. 4N upwards at 11m to right of O.

Miscellaneous Exercises 3

1. 128.7Nm, 91.93N perpendicular to AB.

2. 240N, 460N

3. 64N, 220N, 300N

4. 4m from A

5. 4.35 kNm

6. 1421N, 1274N

7. 857.5N, 1347.5N

8. 2W, 6W, $\frac{4W}{3}$, $\frac{16W}{3}$

9(a). 7.5kN, 10.5kN

(b). 7.5kN

10. 75 tonne, 3.75m

11(a). 105N

(b). 2.91m

(c). 105N, 210N

12. 5kg, $\frac{12a}{5}$

13. R=1N, S=3.5N, P=4.5N

14. 18.4°

15. $\frac{7Wa}{4a-x}$, $\frac{Wa}{2a-x}$

Exercises 4.1

1. 2

2. 3, -13.5

3. 14

4. 27.5

5. 2.5

6. 2

7. 2

8. $\frac{10}{3}$

9. 4

10. 1.6, 2

11. u=12, a=0.5, u=13, a=0

12. t=6

13. t=3, 6

14. $\frac{20}{3}$ ms^{-2}, 4ms^{-2}, 280m

15. 32s, 12s, 16s

16. 5s

17. 15ms^{-1}, 0.15ms^{-2}

18. 10ms^{-1}, 300s

19. 21s

Exercises 4.2

1. 20.4m, 4.1s

2. 29.7, 2.02s

3(a). 1.02s, 8.16s (b). 6.2s, 2.9s

4. 39.4

5. 6.1s

6. 14.9ms^{-1}, 1s

7. 432.9m

8. $\frac{u}{g} + \frac{1}{2}T$

9. 13.5ms^{-1}, 2.3m

10. 22.5m

11(i) 2s (ii) 4.4m from base

12. 7.35m

Miscellaneous Exercises 4

1(a). $1.1ms^{-1}$ (b). 0.95m

(c). $8.2 - 1.2t - 0.2t^2$ (d). 3, 5

2. $\frac{47}{3}$, $\frac{40}{3}$ m

3. 905.6m

4. 126m

5(i). 16 (ii). $0.8ms^{-2}$

6(i). $20ms^{-1}$ (ii) $0.25ms^{-2}$, 2200m

7. $\frac{192}{u+6}$ s, $\frac{60}{u}$ s

(i). 10 (ii). $\frac{1}{3}ms^{-2}$, $\frac{5}{3}ms^{-2}$

9. $\frac{10}{u}$, $\frac{14}{u}$, $\frac{6}{u}$

(i). 5 (ii). $\frac{5}{2}ms^{-2}$, $\frac{25}{6}ms^{-2}$

10(i). $\frac{1}{2}ft^2 + b - ut$

(ii). $-\frac{v^2}{2f} + (v-u)t + b$, $b - \frac{u^2}{2f}$

11. 225s

12(b). $0.4ms^{-2}$, 1125m

(c). 825s

(d). 960s

13. 1m

14. 6, $\frac{2}{3}$

15(i). 11.025m (ii) 1.5s

(iii). $4.9ms^{-1}$

16(i). 0.5m (ii) 1s

17. $\frac{351u^2}{800g}$, $\frac{5u}{4}$

18. $19.6t - 4.9t^2$,

$19.6(t-2) - 4.9(t-2)^2$, 3, $9.8ms^{-1}$

19. 8cm

Exercises 5.1

1. 1800N

2. $2ms^{-2}$

3 (i). 45 (ii). 60m

4. 134

5(i). $3ms^{-2}$ (ii). 1300kg (iii). 96m

6(i). 313.6N (ii) 313.6N

7. 103N, 98N, 95.5N

8. 1404N, 1420N, 1388N

9. 3

10. 0.057

11. $10.2ms^{-2}$

12. g

13. 18.375m, 0.23

14. $2.12ms^{-2}$

15. At 53.1° to force of magnitude 30N and of magnitude $100ms^{-2}$

Exercises 5.2

1. $0.25ms^{-2}$

2(i). 1260N (ii). 3180N

3(i). 1560N (ii). 520N

4. 1000N, 500N, 700N

5(a). (i). 2400N (ii). 1600N (iii). 800N

(b). (i) 17400N (ii). 11600N (iii). 5800N

6. $0.9ms^2$, 385N

Exercises 5.3

1. $2.45ms^{-2}$, 36.75N

2. $3.92\ ms^{-2}$, 41.2N

3. 0.2g, 4.8Mg

4. $6.1\ ms^{-2}$, 18.4N

5. 11.1N

6. 29.4N

7. $6.9ms^{-2}$

8. 33.1N

9. $6.3ms^{-2}$

10. $2.45ms^{-2}$

Miscellaneous Exercises 5

1. 4900N, 13300N

2. 720, 50s

3. $\frac{10}{3}$ s, $\frac{130}{3}$ m

4(i). $\frac{1}{8}ms^{-2}$, $\frac{1}{8}ms^{-2}$

(ii). 198.5N, 196N, 193.5N

5. 552N, $0.65\ ms^{-2}$, $0.65ms^{-2}$

6. 1.65kg, $0.89ms^{-2}$

7. $v^2 > 2a\left(\frac{R}{m} + g\right)$

8(a). 470.4N

(b). 494.9N, 382N

9(a). (i) $1ms^{-2}$ (ii) 1250m

(b). (i) $\frac{5}{3}\ ms^{-2}$ (ii) 12kN

10. 0.37

11. (i) $\frac{3g}{5}$ (ii) $\frac{16mg}{5}$

12. N = 7, T = 38016

13(a). (i) light (ii) smooth, (b) $\frac{3g}{5}$

15. 6mg, 5a

16(a). $\frac{2g}{3}$, $\frac{2mg}{3}$ (b). $\frac{4g}{9}$, $\frac{7mg}{9}$

17. $\frac{3g}{10}$, 2.6mg, $\frac{2d}{5}$

Exercises 6.1

1(a). 0.12Ns

(b). 7.5Ns

(c). 30000Ns

2. 1 Ns

3. -9.6Ns

4. -4.4Ns

5. $4\ ms^{-1}$

6. 10500Ns

7. $2.86ms^{-1}$

8. 1.44 Ns

9. 5.7N

10. 3.4Ns

Exercises 6.2

1. 2

2. 1.25

3. 3.71

4. 3.29

5. 2.2

6. 2.6

7. $8.3ms^{-1}$

8. $0.86ms^{-1}$

9. $2.3kmh^{-1}$

10. $3.79ms^{-1v}$

11. 0.038

Exercises 6.3

1. 0.8m

2. 0.79

3. 1.13m

4. 1.92Ns

5. $4.23ms^{-1}$

6. 4.25, 6.25

7. 0.25, 4

8. 4.29, 0.43

9. 1.8, 3.8

10. 0.47, 3.6

11. 9, 0.17

12. $e > \frac{1}{2}$

13. $0.11\ ms^{-1}$, $0.12ms^{-1}$, $0.77ms^{-1}$

Miscellaneous Exercises 6

1. $9.9ms^{-1}$, $21.8ms^{-1}$
2. $1.15ms^{-1}$, 1385Ns
3. $23.2ms^{-1}$, 8200Nsv
4. $\frac{g}{9}$, $40\frac{mg}{9}$, $\frac{9V}{11}$
5(a). $3.4ms^{-1}$
(b). 0.5
6. $\frac{20u}{3}$, $\frac{2}{3}$
8. 1.06m
9(a). $u|4a-2|$, $u|2a-3|$
(b). $\frac{6a}{5}-1$
(c). $\frac{5}{6} \le a \le \frac{5}{3}$
10. $\frac{5u}{2}$, $\frac{7}{8}$
11. $q > 20$
12. 8mu, 3.33, 0.19
13(a). 2.8u
(b). 6mu
(c). 0.91
14. $\frac{1.2u}{k+1}$, $\frac{(1-0.2k)u}{1+k}$
(i)(a). 2m (b). 0.75m (c). 6.2
15(i). $15ms^{-1}$ (ii) 3.2Ns

Exercises 7.1

1(a). (5, 3)
(b). (1, 3.2)
(c). (0.1, 3.2)
(d). $(-\frac{1}{5}, -\frac{1}{6})$
2. $9\frac{12}{17}$ cm
3. $\frac{4}{3}$m, $\frac{3}{4}$m
4. 5m, 3m
5. m = 4kg, m = 33kg

Exercises 7.2

1(a). (1, 2)
(b). $(3, 1\frac{1}{2})$
(c). $(\frac{8}{3}, \frac{4}{3})$
(d). (1, 0)
(e). $(4\frac{1}{2}, 3)$
(f). $(3, \frac{31}{11})$v
(g). $(3\frac{2}{3}, 3\frac{2}{3})$v
(h). (4, 2.85)
2. 11cm from DC, 12cm from AD
3. $4\frac{2}{3}$ cm

Exercises 7.3

1. 42°
2. $\frac{1}{8}$, $\frac{2}{3}$
3. $\frac{1}{3}$, $\frac{5}{2}$
4. $3\frac{11}{16}$, 59.7°
5. $\frac{1}{10}$

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